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Compactifications and Representations of Transformation Semigroups

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Abstract

This thesis deals essentially (but not from all aspects) with the extension of the notion of semigroup compactification and the construction of a general theory of semitopological nonaffine (affine) transformation semigroup compactifications. It determines those compactifications which are universal with respect to some algebraic or topological properties. As an application of the theory, it is investigated the inclusion relations between function spaces defined on transformation semigroups.

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Preface:

In semigroup theory, it is well known that the study of the functional analytic and dynamical theory of continuous representations of semitopological semigroups is facilitated by the use of semigroup compactifications.

Throughout the thesis, the underlying structure is an abstract transformation semigroup by which we mean a pair consisting of a semigroup and a set, of course together with a semigroup action of the first on the second one. It is called semitopological whenever both the semigroup and the set are endowed with the topologies such that both the semigroup multiplication and the semigroup action are separately continuous.

As the main theme of this thesis, we extend the notion of semigroup compactification and obtain a general theory of compactifications of semitopological transformation semigroups. We then introduce some applications of these compactifications to spaces of functions induced by transformation semigroups. The general approach to this theory (which, for semigroup case, whose seminal technique is due to Loomis [36]) is based on the Gelfand-Naimark theory of commutative C*-algebras. In this way, compactifications of a semitopological transformation semigroup (S, X) appear as the pairs of the spectra of certain C*-algebras of functions on S and X respectively, such that the first spectrum as a semigroup acts on the second one. Therefore first step for constructing this compactifications is to give two Arens type products corresponding to the certain pairs of the duals of function spaces induced by transformation semigroups.

The thesis contains seven chapter, with an epilogue which summarizes inclusion relationships between the relevant function spaces and gives some open questions. Every chapter contains several sections, and begins with a brief introduction, which describes the topics included. For notation and terminology, we shall follow Berglund et al. [2], as much as possible. The background for undefined concepts and unproved statements can be found in Kelley [32], Simmons [49], Rudin [46], Kelley and Namioka [33], Dunford and Schwartz [12], and Berglund et al. [2].

In the first chapter we give a general theory of transformation semigroups first without and then with topology, including the important notion of sub-transformation semigroups, by generalizing the main definitions and results of the classical semigroup theory, and we obtain the elementary essential tools. We introduce the notions of left, right, and two-sided transformation ideals in transformation semigroups, and then study their algebraic-topological properties. We obtain some characterizations of the notions: the maximal sub-transformation group corresponding to a minimal idempotent of the phase semigroup, minimal left (resp. right, two-sided) transformation ideal. We determine the structure of the minimal transformation ideal of a transformation semigroup,in terms of minimal left or right transformation ideals. For example, among the other results, we show that the minimal transformation ideal of a transformation semigroup (S, X) (if there exists) is a transformation group if and only if (S, X) has a unique minimal left transformation ideal and a unique minimal right transformation ideal, which is a generalized form of this fact that the minimal ideal of a semigroup S (if there exists) is a group if and only if S has a unique minimal left ideal and a unique minimal right ideal.

Chapter II provides some ways for constructing new transformation semigroups from old. Here we give the basic properties of transformation semigroups induced by means on certain function spaces. In this chapter, we first extend the notions of introversion and m-introversion to the class of transformation semigroups by introducing the left and right introversion type operators on suitable function spaces on a set under the action of a semigroup, we then define the left and right admissible pairs of special function spaces and the left and right m-admissible pairs of special function algebras, by which we can define one Arens type product for a left admissible pair of suitable function spaces and another one for a right admissible pair.

In chapter III, we first study the spaces of almost periodic and weakly almost periodic functions, and then investigate the close connection between regularity of Arens type products and weak almost periodicity. Also, we define and study some other important classes of function spaces on transformation semigroups.

The general theory of non-affine and affine transformation semigroup compactifications is presented in chapter IV. The main result shows that, for a semitopological transformation semigroup (S, X), there is a one-to-one correspondence between right (resp. left) topological compactifications of (S, X) and left (resp. right) m-admissible pairs of subalgebras, of $\mathcal{C}(S)$ and $\mathcal{C}(X)$ respectively, for (S, X), and also a one-to-one correspondence between right (resp. left) topological affine compactifications of (S, X) and left (resp. right) admissible pairs of subspaces, of $\mathcal{C}(S)$ and $\mathcal{C}(X)$ respectively, for (S, X). We then generalize the theory of subdirect products of (non-affine and affine) compactifications from semigroups to transformation semigroups, and so we can construct the universal compactifications of transformation semigroups in terms of function algebras. In the final section, we extend the notions of right (and left) distal functions defined on a semigroup, and define them on a topological space under the action of a semigroup, and study their properties. We then characterize (in terms of these function spaces) the universal right topological, left topological, and topological transformation group compactifications of a semitopological transformation semigroup.

In chapter V, we study inclusion relationships between function spaces defined on transformation semigroups and obtain some results. For example, we prove that if there exists a separately continuous action of a topologically right simple semitopological semigroup S on a topological space X and if S acts topologically surjectively on X then each weakly almost periodic function on X, with respect to S, is left norm continuous. We also prove that under some translation invariance conditions, each continuous function vanishing at infinity defined on a locally compact non-compact topological space X is weakly almost periodic with respect to a separately continuous action of a locally compact non-compact semitopological semigroup S.

In chapter VI, we define quotient transformation semigroups by extending the notion of congruence, and then show that for a locally compact Hausdorff topological transformation group, the quotients of some certain transformation semigroup compactifications are topologically isomorphic to compactifications of quotients.

In chapter VII, we study representations of transformation semigroups.

In terms of coefficients of these representations, we define and study the space of strongly almost periodic functions on transformation semigroups. we then characterize the universal topological transformation group compactification of a transformation semigroup. We also extend the notion of semigroups of operators to the class of transformation semigroups, and so we introduce transformation semigroups of operators on Banach spaces and study their compactness properties. In particular, we study compactifications of the almost and weakly almost periodic type of them.

In the end, by taking the opportunity, I wish to express my profound gratitude and appreciation to my supervisor, Professor M.A. Pourabdollah, for his unlimited help, valuable guidence and repetitious encouragement during the period of my reseach and preparing the thesis. I also would like to express my deepest thanks to Professor A. Niknam and Dr. H.R. Ebrahimi-vishki for the valuable suggestions and criticisms they have made.

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Contents:

Chapter I

Abstract transformation semigroups

In this chapter, we give a general theory of abstract transformation semigroups. The first section deals with the algebraic theory of abstract transformation semigroups only. We define and study the notions of sub-transformat ion semigroup, the maximal sub-transformation group corresponding to a minimal idempotent of the phase semigroup, left (resp. right, two-sided) transformation ideal, transformation left (resp. right) group. In the second section, we introduce the notion of minimality for left (resp. right, two-sided) transformation ideals, and obtain some characterizations of them. For example, among the other results, we determine the structure of the minimal transformation ideal of a transformation semigroup, in terms of minimal left or right transformation ideals. We also show that the minimal transformation ideal of a transformation semigroup (S, X) (if exists) is a transformation group if and only if (S, X) has a unique minimal left transformation ideal and a unique minimal right transformation ideal. This is a generalized form of the fact that the minimal ideal of a semigroup S (if exists) is a group if and only if S has a unique minimal left ideal and a unique minimal right ideal. In the final section, we study algebraico-topological properties of a transformation semigoup and its sub-structures.

1 Basic concepts

1.1 Definition.

Let S be a semigroup, and let X be a set. The pair (S, X) is called a transformation semigroup, if there is an action of S on X, that is a mapping

$(s, x) \to sx : S \times X \to X$ such that $s(tx) = (st)x$ for all $s, t \in S$ and $x \in X$. S is called the phase semigroup and X the phase set of (S, X). For each $x \in X$, Sx is called an S-orbit of points of X. If $Sx = \{x\}$, then x is called a fixed point of X for S. When S is a group with identity e and $ex = x$ for all $x \in X$, (S, X) is a transformation group.

1.2 Examples.

(a) Every semigroup S is a transformation semigroup (S, S) where the action of S on S is the semigroup multiplication.

(b) Let X be a nonempty set and let S be the semigroup X^X (under composition) of all mappings from X into X. Then (S, X) with the natural action $(s, x) \to s(x)$ is a transformation semigroup. In particular if G is the subgroup of X^X consisting of all one-to-one mappings from X onto X, then (G, X) is a sub-transformation semigroup of (S, X), as definition in 1.3(a).

(c) Let X be a nonempty set and let $\mathbb{N}$ be the additive semigroup of natural numbers, then for each map $f: X \to X$, $(\mathbb{N}, X)$ under the action $(n, x) \to f^n(x)$ is a transformation semigroup. If f is one-to-one and onto then the additive group $\mathbb{Z}$ of integers acts on X in the same way.

(d) Let (S, X) be a transformation semigroup, and let A be a nonempty set. Consider the direct product semigroup $T := S^A$, and the cartesian product set $Y := X^A$. Then (T, Y) with the action $(t, y) \to ty : T \times Y \to Y$ defined by $ty(a) = t(a)y(a)$ for $a \in A$ is a transformation semigroup.

(e) Let (S, X) be a transformation semigroup where S is abelian, and denote the set of all mappings from X into a nonempty set Y by Y^X. Then (S, Y^X) under the action $(s, f) \to sf : S \times Y^X \to Y^X$ defined by $sf(x) = f(sx)$ for $x \in X$ is a transformation semigroup.

(f) Let (Y, Z) be a pair of linear spaces. Let S be the semigroup (under composition) of linear operators on Y, and let X be the space of linear operators from Z into Y. Then (S, X) with the action $(s, x) \to sx : S \times X \to X$ defined by $(sx)(z) = s(x(z))$ for $z \in Z$ is a transformation semigroup.

For $s, t \in S$, $x \in X$ the translation mappings $\lambda_s: S \to S$, $\rho_t: S \to S$, $\dot{\lambda}_s: X \to X$ and $\dot{\rho}_x: S \to X$ are defined by $\lambda_s(t) = st = \rho_t(s)$ and $\dot{\lambda}_s(x) = sx = \dot{\rho}_x(s)$.

For $s \in S$, $x \in X$, $T, T' \subseteq S$ and $Y \subseteq X$ we define

$$Ts = \rho_s(T)\ ,\ sT = \lambda_s(T)\ ,\ Ts^{-1} = \rho_s^{-1}(T)\ ,\ s^{-1}T = \lambda_s^{-1}(T)$$

$$Tx = \dot{\rho}_x(T) \ , \ sY = \dot{\lambda}_s(Y) \ , s^{-1}Y = \dot{\lambda}_s^{-1}(Y) \ , \ Yx^{-1} = \dot{\rho}_x^{-1}(Y)$$
$$TT' = \cup_{t\in T}\, tT' = \cup_{t'\in T'} Tt' \ , \ TY = \cup_{y\in Y} Ty = \cup_{t\in T}\, tY$$

We say Y is T-invariant if $TY \subseteq Y$. Clearly the union and the nonempty intersection of two nonempty T-invariant subsets is T-invariant. Y is minimal T-invariant if it properly contains no T-invariant subsets of X. By definition, Y is minimal T-invariant if and only if $Y = Ty$ for all $y\in Y$. Now

1.3 Definition.

We call (T, Y),

(a) a *sub-transformation semigroup* of (S, X) if $T^2 \subseteq T$ and $TY \subseteq Y$, that is if T is a subsemigroup of S and Y is a T-invariant subset of X. Furthermore if T is a subgroup of S with identity e and $ey = y$ for each $y\in Y$, then (T, Y) is called a sub-transformation group of (S, X);

(b) a *left transformation ideal* of (S, X) if $ST \subseteq T$ and $SY \subseteq Y$, that is if T is a left ideal of S and Y is an S-invariant subset of X;

(c) a *right transformation ideal* of (S, X) if $TS \subseteq T$ and $TX \subseteq Y$, that is if T is a right ideal of S and T retracts X into Y;

(d) a *(two-sided)transformation ideal* of (S, X) if (T, Y) is both a left transformation ideal and a right transformation ideal.

In all of these definitions we say (S, X) contains (T, Y) and we write $(T, Y) \subseteq (S, X)$. If $T \neq S$ or $Y \neq X$, we say (T, Y) is proper.

1.4 Examples.

(a) By definition, each left or right transformation ideal of (S, X) is a sub-transformation semigroup of (S, X). Also if (T, Y) is a right transformation ideal of (S, X) and Y' is a subset of X containing Y, then (T, Y') is also a right transformation ideal of (S, X).

(b) Let (S, X) be a transformation semigroup. If T is a subsemigroup (resp. left ideal, right ideal, ideal) of S, then (T, X), (T, SX) and (T, TX) are sub-transformation semigroups (resp. left transformation ideals, right transformation ideals, transformation ideals) of (S, X). But if T is a subgroup of S then (T, X) need not be a sub-transformation group of (S, X).

(c) Let (S, X) be a transformation semigroup. If Y is an S-invariant subset of X, then (S, Y) is a left transformation ideal of (S, X). In particular if S retracts X into Y, then (S,Y) is a two-sided transformation ideal of (S,X).

1.5 Proposition.

Let (S,X) be a transformation semigroup.

(i) *If $\{(T_i,Y_i) : i\in I\}$ is a family of sub-transformation semigroups (resp. left transformation ideals, right transformation ideals, transformation ideals) of (S,X). Then the intersection defined by*

$$\cap_{i\in I}(T_i,Y_i) := (\cap_{i\in I}T_i, \cap_{i\in I}Y_i)$$

of all members of the family is also a sub-transformation semigroup (resp. left transformation ideal, right transformation ideal, transformation ideal) of (S,X), provided that $\cap_{i\in I}T_i$ and $\cap_{i\in I}Y_i$ are nonempty.

(ii) *If $\{(T_i,Y_i) : i\in I\}$ is a family of left (resp. right, two-sided) transformation ideals of (S,X). Then the union defined by*

$$\cup_{i\in I}(T_i,Y_i) := (\cup_{i\in I}T_i, \cup_{i\in I}Y_i)$$

of all members of the family is also a left (resp. right, two-sided) transformation ideal of (S,X).

Proof. The conclusions follow from the relations

$$(\cap T_i)(\cap Y_i) \subseteq \cap(T_iY_i) \subseteq \cap Y_i,$$
$$S(\cap Y_i) = \cap(SY_i) \subseteq \cap Y_i, \quad (\cap T_i)X = \cap(T_iX) \subseteq \cap Y_i,$$
$$S(\cup Y_i) = \cup(SY_i), \quad (\cup T_i)X = \cup(T_iX). \quad \square$$

1.6 Proposition.

Let (S,X) be a transformation semigroup and e an idempotent in S. Let $\{(T_i,Y_i) : i\in I\}$ be the family of all sub-transformation groups of (S,X) such that each T_i contains e. Then the union, defined by $\cup_{i\in I}(T_i,Y_i) := (\cup_{i\in I}T_i, \cup_{i\in I}Y_i)$ of all members of the family is a sub-transformation group of (S,X), which is called the maximal sub-transformation group of (S,X) corresponding to $e\in E(S)$.

Proof. $\cup_{i \in I} T_i$ is the maximal subgroup $H(e)$ of S containing e. Verifying that $(\cup_{i \in I} T_i, \cup_{i \in I} Y_i)$ is a sub-transformation semigroup of (S, X) is easy. Finally for each $y \in \cup_{i \in I} Y_i$ we have $ey = y$. □

1.7 Corollary.

The maximal sub-transformation group of (S, X) corresponding to $e \in E(S)$ is $(H(e), H(e)X)$, where $H(e)$ is the maximal subgroup of S containing e.

1.8 Definition.

Let (S, X) be a transformation semigroup. (S, X) is called *left* (resp. *right*) *transformation simple* if it has no proper left (resp. right) transformation ideals. (S, X) is *transformation simple* if it has no proper two-sided transformation ideals.

Clearly, if (S, X) is left or right transformation simple then it is transformation simple.

We say S is

(a) *transitive* (resp. *point transitive*) on X if $Sx = X$ for each (resp. for some) $x \in X$,

(b) *effective* (resp. *strongly effective*) on X if $s, t \in S$ with $s \neq t$ implies $sx \neq tx$ for some (resp. for all) $x \in X$,

(c) *surjective* (resp. *point surjective*) on X if $sX = X$ for all (resp. for some) $s \in S$,

(d) *injective* on X if $x, y \in X$ with $x \neq y$ implies $sx \neq sy$ for all $s \in S$.

We say (S, X) is a transitive (resp. effective, strongly effective, surjective, injective) transformation semigroup if S is transitive (resp. effective, strongly effective, surjective, injective) on X.

The next result characterizes left, right, and two-sided transformation simple transformation semigroups:

1.9 Proposition.

A transformation semigroup (S, X) is

(i) *left transformation simple if and only if S is left simple and S is transitive on X,*

(ii) *right transformation simple if and only if S is right simple and S is surjective on X,*

(iii) *transformation simple if and only if S is simple and $SsX = X$ for all $s \in S$.*

Proof. (i) The necessity is clear, since (Ss, Sx) is a left transformation ideal. For the sufficiency observe that if (T, Y) is a left transformation ideal and $(t, y) \in T \times Y$, then $(S, X) = (St, Sy) \subseteq (T, Y)$. For (ii) and (iii) argue similar to (i). □

1.10 Lemma.

Let (S, X) be a transformation semigroup. For an idempotent e in S, the following are equivalent.

(i) $eX = X$.

(ii) $ex = x$ *for all* $x \in X$.

(iii) $ex = ey$ *implies* $x = y$ *for all* $x, y \in X$.

Proof. (i) implies (ii). For each $x \in X$, there is a $y \in Y$ such that $x = ey = e(ey) = ex$. (ii) obviously implies (i) and (iii).

(iii) implies (ii). Let $x \in X$ and $x \neq ex$, then $ex \neq e(ex) = ex$ which is impossible. □

1.11 Proposition.

Let (S, X) be a transformation semigroup.

(i) *If S is left simple, then S-orbits of points of X are either disjoint or identical.*

(ii) *If $Ss = S$ and $sX = X$ for some $s \in S$, then $x \in Sx$ for all $x \in X$.*

(iii) *If S is left simple and $sX = X$ for some $s \in S$, then X is a disjoint union of S-orbits. In this case S is transitive on X if and only if S is point transitive on X.*

(iv) *If S is right simple, then $sX = tX$ for all $s, t \in S$.*

(v) *If S is right simple and S is point transitive or point surjective on X, then S is surjective on X.*

(vi) *If (S, X) is a transformation group, then S is surjective and injective on X. Also transitivity and point transitivity are equivalent.*

Proof. **(i)** Let S be left simple, $x, y \in X$ and $y \in Sx$, then $y = sx$ for some $s \in S$. Thus $Sy = Ssx = Sx$.

(ii) Let $Ss = S$ and $sX = X$ for some $s \in S$ and let $x \in X$, then $x = sy$ for some $y \in X$ and so $x \in Sy$. But $Sx = Ssy = Sy$.

(iii) follows from (i) and (ii).

(iv) Let S be right simple and $s, t \in S, x \in X$, then $sx = ts'x$ for some $s' \in S$ and so $sX \subseteq tX$. Similarly $tX \subseteq sX$.

(v) For each $s \in S$ we have $sX = sSx_\circ = Sx_\circ = X$, where $x_\circ$ is a transitive point.

(vi) Let (S, X) be a transformation group and let $s \in S, x \in X$, then $x = ex = ss^{-1}x \in sX$ and so $sX = X$. If $y \in X$ and $sx = sy$, then $x = ex = s^{-1}sx = s^{-1}sy = ey = y$ where e is the identity of S. Second assertion follows from (iii). □

1.12 Theorem.

A transformation semigroup (S, X) is a transitive transformation group if and only if (S, X) is left transformation simple and right transformation simple.

Proof. Let (S, X) be a transitive transformation group, then S is left simple and right simple, and so (S, X) is left transformation simple and right transformation simple (by Proposition 1.9 and Proposition 1.11(vi)). The converse of the conclusion follows from Proposition 1.9 and [2; 1.1.17]. □

Recall, say from [2; 1.2.18], that a semigroup S is called a left (resp. right) group if for each pair $s, t \in S$, there is a unique $u \in S$ such that $us = t$ (resp. $su = t$). Now we extend these notions.

1.13 Definition.

A transformation semigroup (S, X) is called a *transformation left* (resp. *right*) *group* if S is a left (resp. right) group and for each pair $x \in X$ (resp.

$s \in S$) and $y \in X$ there is a unique $s \in S$ (resp. $x \in X$) such that $sx = y$.

The following proposition characterizes transformation left groups and transformation right groups.

1.14 Proposition.

Let (S, X) be a transformation semigroup.
(i) *The following are equivalent.*
(a) *(S, X) is left transformation simple and S is right cancellative and S is strongely effective on X.*
(b) *S is a left group and S is transitive on X and S is strongely effective on X.*
(c) *(S, X) is a transformation left group.*
(ii) *The following are equivalent.*
(a) *(S, X) is right transformation simple and S is left cancellative and S is injective on X.*
(b) *S is a right group and S is surjective on X and S is injective on X.*
(c) *(S, X) is a transformation right group.*

Proof. **(i)** (a) implies (b). If (a) holds, then S is left simple and right cancellative and so S is a left group, by [2;1.2.19]. Also, S is transitive on X, by Proposition 1.9(i).

(b) implies (c). If (b) holds then for each $x, y \in X$, since $Sx = X$ there is an $s \in S$ such that $sx = y$. This s is unique since S is strongly effective on X.

(c) implies (a). If (c) holds, then S is a left group and so S is left simple and right cancellative, by [2; 1.2.19]. To prove the transitivity of S on X, for any $x \in X$ we must show that $Sx = X$ or $X \subseteq Sx$. Let $y \in X$ then by assumption, there is a unique $s \in S$ such that $y = sx \in Sx$. The uniqueness of s implies that S is strongly effective on X.

(ii) (a) implies (b). If (a) holds, then S is right simple and left cancellative and so S is a right group. Also, S is sujective and injective on X.

(b) obviously implies (c).

(c) implies (a). If (c) holds, then S is a right group and so S is right simple and left cancellative, by dual of [2;1.2.19]. To prove the surjectivity of S on X, for any $s \in S$ we must show that $sX = X$ or $X \subseteq sX$. Let $y \in X$ then,