Abstract

THE ZEROS OF SOME GOLAY POLYNOMIALS

By

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We have seen the use of polynomials in the various areas of mathematics and other sciences, in particular, those polynomials that their coefficients are restricted with special conditions. There are not any general formulas regarding the zeros of polynomials of degree more than four. So two important questions about polynomials arise. The first is the problem of finding bounds for the zeros of polynomials and the second is the number of the zeros of a polynomial in an open disk. In this thesis, we first focus on the problem of bounding the zeros of complex polynomials when their coefficients are restricted in various types. Among them those polynomials that have coefficients in the closure of the unit disk $\mathbb{T}$, in particular, the Rudin-Shapiro polynomials, we introduce an annulus containing all the zeros.

Finally, we look at the problem of finding upper bounds for the number of zeros of some other type of complex polynomials that lie in an open disk centered at $z_0 \in \mathbb{C}$ with radius $r > 0$. All methods are used in this work are analytic with tools in complex analysis.

Contents

Introduction

The theory of finding bounds for the zeros of polynomials and the location of zeros of polynomials has applications in several areas of contemporary applied mathematics including root approximation, coding theory, electrical networks, linear control systems, and signal processing. Because of these applications, there is a need for obtaining better and better results in these matters. Gauss and Cauchy were the earliest contributors in the study of these subjects, and since then these subjects have been studied by many people. This thesis is devoted to the following problems:

i) Finding bounds for the zeros and the location of the zeros of polynomials.

ii) Providing upper bounds for the number of the zeros of polynomials in an open disk, in particular, polynomials which have coefficients in a finite subset of the complex plane.

The first chapter, we begin with an introduction to the theory of polynomials and some of their properties.

In chapter two, we introduce a special type of trigonometric polynomials called the Rudin-Shapiro polynomials and then we provide some basic properties of them.

In chapter three, we present an annulus that contains all the zeros of Littlewood polynomials and we prove results on bounds for the zeros and also the location of zeros of complex polynomials which their coefficients are restricted with special conditions such as the polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ whenever $|a_i| \leq |a_n|$ for all $i = 0, \cdots, n-1$ and the monic polynomial $p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ whenever $|a_i| \leq |a_0|$, for every $i = 1, \cdots, n-1$.

At the end, we look at the problem of finding the number of the zeros in an open disk of complex polynomials which their coefficients are in the closed unit disk.

Chapter 1

Introduction to Polynomials

1 Introduction to Polynomials

Polynomials pervade mathematics, and much that is beautiful in mathematics is related to polynomials. Virtually every branch of mathematics, from algebraic number theory and algebraic geometry to applied analysis, Fourier analysis, and computer sciences, has its corpus of theory arising from the study of polynomials. This chapter serves as a general introduction to the theory of polynomials and the necessary preliminaries for our work. Also in this chapter, we state and prove some basic properties of polynomials. Undoubtedly, the most basic and important theorem concerning polynomials is the Fundamental Theorem of Algebra. This theorem, in fact, tells us that every polynomial factors completely over the complex numbers. There are many proofs of this theorem based on elementary properties of complex functions. The focus for this thesis is the polynomial of a single variable. Highlights of this chapter include: Descartes' rules of sign, Sturm's Theorem, Rouché's Theorem, Fundamental Theorem of Algebra, and Cauchy's Theorem.

1.1 Preliminaries definitions and notations

Definition 1.1.1. A function of a single variable t is a polynomial on its domain if we can put it in the form

$$p(t) := a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0,$$

where $a_n, a_{n-1}, \cdots a_1, a_0$ are constants.

The constants $a_n, a_{n-1}, \cdots a_1, a_0$ are called the coefficients of the polynomial

$p(t)$ and the nonnegative integer n is called degree of $p(t)$ if $n \neq 0$.

A *zero (root)* of a polynomial $p(t)$ is any number r for which $p(r) = 0$.

Throughout this thesis we use the following notations.

The symbols $\mathbb{R}$ and $\mathbb{C}$ use for the set of real and complex numbers, and also $\mathbb{R}[x]$ and $\mathbb{C}[z]$ use for the set of all real polynomials with coefficients in $\mathbb{R}$ and all complex polynomials with coefficients in $\mathbb{C}$, respectively. The notation $\mathbb{T}$ will denote the unit circle in the complex plane which can be identified with the interval $[-\pi, \pi)$. Finally, $D(z_0, r)$ is used for the open disk in the complex plane centered at the point z_0 with radius $r > 0$.

Definition 1.1.2. The polynomial $p(t) = a_n t^n + a_{n-1} t^{n-1} + ... + a_1 t + a_0, \ a_k \in \mathbb{C}$ and $a_n \neq 0$ is called *monic* if its leading coefficient a_n equals 1.

Definition 1.1.3. A trigonometric polynomial of degree n is any function defined on $\mathbb{R}$ by

$$P_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos(kt) + b_k \sin(kt)\right),$$

where a_k's and b_k's are complex numbers.

Because of Euler's formulas we also can write this as follows:

$$P_n(t) = \sum_{k=-n}^{n} c_k e^{ikt},$$

where for every $0 \leq k \leq n$ (letting $b_0 = 0$)

$$c_k = (a_k - ib_k)/2, \quad c_{-k} = (a_k + ib_k)/2.$$

Obviously trigonometric polynomials are 2π-periodic functions. Moreover if a sequence of trigonometric polynomials converges pointwise to a complex-valued function f on $\mathbb{R}$, then f is also 2π-periodic, i.e. $f(t + 2\pi) = f(t)$ for all $t \in \mathbb{R}$.

1.2 A short history of polynomials

Historically, questions relating to polynomials, for example, the solution of polynomial equations, gave rise to some of the most important problems of the day. The subject is now much too large to attempt an encyclopedic coverage. The importance of the solution of polynomial equations in the history of mathematics is hard to overestimate. The Greeks of the classical period understood quadratic equations (at least when both roots were positive), but could not solve cubics. The explicit solutions of the cubic and quadratic equations in the sixteenth century were due to Niccolo Tartaglia (ca 1500-1557), Ludovico Ferrari (1522-1565), and Scipione del Ferro (ca 1465-1526) and were popularized by the publication in 1545 of the "Ars Magna "of Girolamo Cardano (1501-1576). The exact priorities are not entirely clear, but del Ferro probably has the strongest claim on the solution of the cubic. These discoveries gave western mathematics an enormous boost in part because they represented one of the first really major improvements on Greek mathematics. The impossibility of finding the zeros of a polynomial of degree at least 5, in general, by a formula containing additions, subtractions, multiplications, divisions, and radicals would await Niels Henrik Abel (1802-1829) in his 1824 publication of "On the Algebraic Resolution of Equations. "Indeed, so much algebra, including Galois theory, analysis, and particularly complex analysis, is born out of these ideas that it is hard to imagine how the flow of mathematics might have proceeded without these issues being raised. For further history, see [11].

1.3 Some basic results on polynomials

There are no general formulas on the zeros of polynomials of degree more than four. So root finding for them resort to numerical methods, but there are closed-form formulas for roots of low order polynomials. Here we do not mention them, for more details for roots of polynomials of degree up to four see [1].

Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n. The Fundamental Theorem of Algebra states that p has n real or complex zeros (counting multiplicities). If the coefficients $a_0, \cdots, a_n$ are real, then the complex zeros occur in conjugate pairs, but if the coefficients are complex, the complex zeros need not be related.

Using *Descartes' rules of sign*, we can count the number of real positive zeros that $p(x)$ has. More specifically, let m_p be the number of variations in the sign of the coefficients $a_n, a_{n-1}, \cdots, a_0$ (ignoring coefficients that are zero), then we have the Descartes' rules of sign as follows:

Theorem 1.3.1. (Descartes' rules of sign) *Let n_p be the number of real positive zeros of the polynomial p and m_p be the number of variations in sign of the coefficients of p. Then*

(i) $n_p \leq m_p$,

(ii) $m_p - n_p$ is an even integer.

Similarly, the number of real negative zeros of $p(x)$ is related to the number of sign changes in the coefficients of $p(-x)$.

Example 1.3.2. Consider the polynomial $p(x) = x^4 + 2x^2 - x - 1$. Since $m_p = 1$, so n_p is either 0 or 1 by rule (i), but by rule (ii) $m_p - n_p$ must be even. Hence $n_p = 1$. Now look at $p(-x) = x^4 + 2x^2 + x - 1$. Again, the coefficients have one variation in sign, so $p(-x)$ has one positive zero. In other words, $p(x)$ has one negative zero.

To summarize, simply by looking at the coefficients, we conclude that $p(x)$ has one positive real zero, one negative real zero, and two complex zeros as a conjugate pair.

Descartes' rules of sign still leaves an uncertainty as to the exact number of real zeros of a polynomial with real coefficients, for example, $p(x) = x^4 - x^3 + x^2 - x + 1$. The problem of finding an exact test for the number of real zeros of a polynomial equation was solved in a surprising simple way in 1829 by the French mathematician Charles Sturm (1803-1855). He showed a method to count the real zeros which lie within any given interval.

Let $f(x)$ be a polynomial in $\mathbb{R}[x]$. Recall that α is a *multiple root* of f if $(x - \alpha)^2$ divides f, otherwise, α is said a *simple root*. If f has a multiple root then we can write $f = (x - \alpha)^k g(x)$ with $(x - \alpha, g) = 1$, then by the product rule we have $(f, f') = (x - \alpha)^{k-1}(g, g')$ and $f/(f, f') = (x - \alpha).g/(g, g')$. Now it follows that α is a simple root of $f/(f, f')$. Moreover as an easy consequence of the product rule is that f has a multiple root if and only if f and f' are not relatively prime.

As a consequence of this, if f and f' are not relatively prime, then $f/(f, f')$ has the same set of zeros as f, but each is a simple root. Therefore, we can then assume that f and f' are relatively prime and that each root of f is simple. The derivative f' then vanishes for none of these roots and $(f, f') = 1$.

Denote $f(x)$ by $f_0(x)$ and its derivative $f'(x)$ by $f_1(x)$ and use the Euclidean Algorithm to find the greatest common divisor of f and f', calling the quotients resulting from the successive divisions q_1, q_2,...,q_{k-1} and the remainders $-f_2$, $-f_3$,...,$-f_k$ (Note the unconventional choice of sign for the remainders!), therefore

$$
\begin{aligned}
f_0(x) &= q_1(x)f_1(x) - f_2(x), \\
f_1(x) &= q_2(x)f_2(x) - f_3(x), \\
&\vdots \\
f_{k-2}(x) &= q_{k-1}(x)f_{k-1}(x) - f_k(x),
\end{aligned}
$$

where f_k is a constant, and for $1 \le i \le k$, $f_i(x)$ is of degree lower than that of $f_{i-1}(x)$.

Note that the last non-vanishing remainder f_k (or f_{k-1} when $f_k = 0$) is the greatest common divisor of $f(x)$ and $f'(x)$, and consequently possesses the same sign over the whole interval.

The sequence $f_0, f_1, \cdots, f_k$ (or f_{k-1} when $f_k = 0$) is called the *Sturm sequence* for the polynomial f and in this connection are called *Sturm functions*.

Theorem 1.3.3. (Sturm's Theorem) *Let $f(x)$ be a polynomial in $\mathbb{R}[x]$. Then the number of distinct real zeros of $f(x)$ in (a, b) is V_a-V_b, where V_c denotes the number of variations in sign of the sequence $f_0(c), f_1(c) \cdots, f_{k-1}(c), f_k$.*

In fact, we can multiply f by a positive constant, or a factor involving x, provided that the factor remains positive throughout (a, b), and the modified function can be used for computing all further terms f_i of the sequence.

Example 1.3.4. Using Sturm's Theorem to isolate the real zeros of the equation

$$x^5 + 5x^4 - 20x^2 - 10x + 2 = 0.$$

We first compute the Sturm functions

$$\begin{aligned} f_0(x) &= x^5 + 5x^4 - 20x^2 - 10x + 2, \\ f_1(x) &= x^4 + 4x^3 - 8x - 2, \\ f_2(x) &= x^3 + 3x^2 - 1, \\ f_3(x) &= 3x^2 + 7x + 1, \\ f_4(x) &= 17x + 11, \\ f_5(x) &= 1. \end{aligned}$$

By setting $x = -\infty, 0, \infty$, we see that there are three negative zeros and two positive zeros. All zeros lie between -10 and 10, in fact, between -5 and 5. We then try all integral values between -5 and 5. The following table records the work:

	$-\infty$	-10	-5	-4	-3	-2	-1	0	1	2	5	10	∞
f_0	-	-	-	-	+	-	-	+	-	+	+	+	+
f_1	+	+	+	+	-	-	-	-	-	+	+	+	+
f_2	-	-	-	-	-	+	+	-	+	+	+	+	+
f_3	+	+	+	+	+	-	-	+	+	+	+	+	+
f_4	-	-	-	-	-	-	-	+	+	+	+	+	+
f_5	+	+	+	+	+	+	+	+	+	+	+	+	+
var.	5	5	5	5	4	3	3	2	1	0	0	0	0

Thus there is a zero in $(-4, -3)$, a zero in $(-3, -2)$, a zero in $(-1, 0)$, a zero in $(0, 1)$, and a zero in $(1, 2)$.

Theorem 1.3.5. (Liouville's Theorem) *A bounded entire function is constant.*

The Fundamental Theorem of Algebra appears to have been given its name by Gauss, although the result was familiar long before; it resisted rigorous proof by d'Alembert (1740), Euler (1749), and Lagrange (1772). It was more commonly formulated as a real theorem, namely; *every real polynomial factors completely into real linear or quadratic factors.* (This is an essential result for the integration of rational functions.) Girard has a claim to priority of formulation. In his "Invention Nouvelle en L' Algebra "of 1629 he wrote "every equation of degree n has as many solutions as the exponent of the highest term."Gauss gave the first satisfactory proof in 1799 in his doctoral dissertation, and he gave three proofs during his lifetime. His first proof was, titled "A new proof that every rational integral function of one variable can be resolved into real factors of the first or second degree", was in fact the first more or less satisfactory proof.

Theorem 1.3.6. (Fundamental Theorem of Algebra) *Every nonconstant polynomial has at least one complex zero.*

This major theorem prove directly from Liouville's Theorem. A very general class of bounds on the magnitude of roots is implied by the Rouché Theorem.

Theorem 1.3.7. (Rouché's Theorem) *Suppose that f and g are analytic inside and on a simple closed path γ (for most purposes we may use γ a circle). If*

$$|f(z) - g(z)| < |f(z)|$$

for every $z \in \gamma$, then f and g have the same number of zeros inside γ (counting multiplicities).

As an application, Rouché's Theorem can be used to give a short proof of the Fundamental Theorem of Algebra as follows:

Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n, choose $R > 1$ so large that:

$$|a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}| \ < \ |a_n|R^n = |a_n z^n| \ for \ \ |z| = R.$$

Since the function $f(z) = a_n z^n$ has exactly n zeros inside the circle $|z| = R$, so by Rouché Theorem $p(z)$ has exactly n zeros inside this circle.

The following theorem is a quantitative version of the Fundamental Theorem of Algebra due to Cauchy [1829]. We offer a proof that does not assume the Fundamental Theorem of Algebra, but does require some elementary complex analysis. The proof can be found in [7].

Theorem 1.3.8. *The polynomial*

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \quad a_k \in \mathbb{C}, \quad a_n \neq 0,$$

has exactly n zeros. These all zeros lie in the open disk centered at the origin with radius r, where

$$r := 1 + \max \frac{|a_k|}{|a_n|}, \quad k = 0, \cdots, n-1.$$

Proof. We may assume that $a_0 \neq 0$, or we may first divide by z^k for some k. Now observe that

$$g(x) := |a_0| + |a_1|x + \cdots + |a_{n-1}|x^{n-1} - |a_n|x^n$$

satisfies $g(0) > 0$ and $\lim_{x\to\infty} g(x) = -\infty$.

So by the Intermediate Value Theorem, g has a zero in $(0, \infty)$. Let s be this zero. Then for $|z| > s$,

$$|p(z) - a_n z^n| \leq |a_0| + |a_1 z| + \cdots + |a_{n-1}z^{n-1}| < |a_n z^n|.$$

Now by Rouché Theorem $p(z)$ and $a_n z^n$ have exactly n zeros in any disk of radius greater than s. It remains to observe that if $x \geq r$, then $g(x) < 0$ and then $s < r$. Indeed for

$$x \geq 1 + \max \frac{|a_k|}{|a_n|}, \quad k = 0, ..., n-1,$$

we have

$$\begin{aligned} g(x) &\leq |a_n|x^n \left(-1 + \left(\max \frac{|a_k|}{|a_n|} \right) \sum_{k=0}^{n-1} x^{k-n} \right) \\ &< |a_n|x^n \left(-1 + \left(\max \frac{|a_k|}{|a_n|} \right) \frac{1}{x-1} \right) \\ &\leq 0 \end{aligned}$$

□

Theorem 1.3.9. (Cauchy's Theorem) *Given a polynomial* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \;\; a_n \neq 0$, *define the polynomials*

$$p(x) = |a_n|x^n - |a_{n-1}|x^{n-1} - \cdots - |a_0|,$$

and

$$q(x) = |a_n|x^n + |a_{n-1}|x^{n-1} + \cdots + |a_1|x - |a_0|.$$

Using Descartes' rules of sign, $p(x)$ *has exactly one real positive zero* R *and* $q(x)$ *has exactly one real positive zero* r. *Then all the zeros of* $f(x)$ *lie in the annulus*

$$r \leq |z| \leq R.$$

We can use the above bounds as heuristics that give us a way of localizing the possible zeros of a polynomial. By localizing the zeros, we can guide the initial guesses of our numerical root finders. The exact relationship between the coefficients of a polynomial and the location of its zeros is very complicated. Of course, the more information we have about the coefficients, the better results we can hope for. The following pretty theorem emphasizes this subject. This theorem presented in [7].

Theorem 1.3.10. (Enestr$\ddot{o}$m - Kakeya). *If*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with

$$a_0 \geq a_1 \geq \cdots \geq a_n > 0,$$

then all the zeros of p lie outside the open unit disk.

Proof. Consider the polynomial

$$(1-z)p(z) = a_0 + (a_1 - a_0)z + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1}.$$

We have

$$|(1-z)p(z)| \geq a_0 - [(a_0 - a_1)|z| + \cdots + (a_{n-1} - a_n)|z|^n + a_n|z|^{n+1}].$$

Since $a_k - a_{k+1} \geq 0$, the right-hand expression above decreases as $|z|$ increases. Thus for $|z| < 1$, we obtain

$$|(1-z)p(z)| > a_0 - [(a_0 - a_1) + \cdots + (a_{n-1} - a_n) + a_n] = 0,$$

and the result follows. □

Corollary 1.3.11. *Suppose that*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

is a polynomial of degree n with $a_k > 0$ for each k. Then all the zeros of p lie in the annulus

$$r_1 \leq |z| \leq r_2,$$

where for $k = 0, \cdots, n-1$,

$$r_1 = \min \frac{a_k}{a_{k+1}} \quad \textit{and} \quad r_2 = \max \frac{a_k}{a_{k+1}}.$$

1.4 Littlewood polynomials

In this section, we define the Littlewood polynomials and state some results about them. In chapter three, we introduce an annulus containing all the zeros of Littlewood polynomials.

Let $\mathcal{U}_n \subset \mathbb{C}[z]$ be the class of all degree n polynomials h

$$h(z) = \sum_{k=0}^{n} a_k z^k,$$

so that $|a_k| = 1$ for all k.

We consider $\mathcal{L}_n \subset \mathcal{U}_n$ so that if $h \in \mathcal{L}_n$, then $a_k = \pm 1$ for all k.

Definition 1.4.1. The members of $\mathcal{U}_n$ and $\mathcal{L}_n$ are called *Unimodular* and *Littlewood* (or sometimes real unimodular) polynomials, respectively.

A special type of Littlewood polynomials are the Rudin-Shapiro polynomials, which we will introduce them in the next chapter.

The next theorem proved in [9] provides upper bounds for the number of real zeros of those polynomials that their coefficients are restricted in the closed unit disk.

Theorem 1.4.2. **(i)** *There is an absolute constant* $c_1 > 0$ *such that every polynomial* p *of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad a_j \in \mathbb{C}, \quad |a_0| = 1, \quad |a_j| \leq 1,$$

has at most $c_1\sqrt{n}$ *zeros in* $[-1,1]$.

(ii) *There is an absolute constant* $c_2 > 0$ *such that every polynomial* p *of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad a_j \in \mathbb{C}, \quad |a_n| = 1, \quad |a_j| \leq 1,$$

has at most $c_2\sqrt{n}$ *zeros in* $\mathbb{R}\backslash(-1,1)$.

(iii) *There is an absolute constant* $c_3 > 0$ *such that every polynomial* p *of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad a_j \in \mathbb{C}, \quad |a_0| = |a_n| = 1, \quad |a_j| \leq 1,$$

has at most $c_3\sqrt{n}$ *real zeros.*

Suppose that p is a complex polynomial of degree n with complex coefficients as

$$p(z) := \sum_{j=0}^{n} a_j z^j.$$

We will consider the following three types of the above polynomial and in chapter three we prove some properties of these types of polynomials.

Type 1. $|a_0| = 1$ and $|a_k| \leq 1$ for every $k \in \{1, 2, \cdots, n\}$.

Type 2. $|a_n| = 1$ and $|a_k| \leq 1$ for every $k \in \{0, 1, \cdots, n-1\}$.

Type 3. $|a_0| = |a_n| = 1$ and $|a_k| \leq 1$ for every $k \in \{1, 2, \cdots, n-1\}$.

Borwein, Erdélyi, and Littmann [10] proved that any polynomial of type 3 has at least $8\sqrt{n}\log n$ zeros in disk with center on the unit circle and radius $33\pi\frac{\log n}{\sqrt{n}}$.

Chapter 2

Rudin-Shapiro Polynomials