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Ph. D. Thesis in Mathematical Statistics

INFINITE DIMENSIONAL GARCH MODELS

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IN THE NAME OF GOD

Declaration Form

I, Zahra Sajjadnia, a Mathematical Statistics student majored in Statistical Inference from the college of Sciences, declare that this thesis is the result of my research and I have written the exact reference and full indication wherever I use others' sources. I also declare that the research and the topic of my thesis are not reduplicative and guarantee that I will not disseminate its accomplishments and not make them accessible to others without the permission of the university. According to the regulations of the mental and spiritual ownership, all rights of this belongs to Shiraz University.

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TO

NUCLEAR MARTYRS OF OUR COUNTRY IRAN,

DR. ALIMOHAMMADI,

DR. SHAHRYARI,

DR. AHMADI-ROSHAN,

DR. REZAYEE-NEJAD,

AND

OTHER MARTYRS OF OUR COUNTRY.

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ABSTRACT

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This thesis has two parts. In the first part, we introduce and study the Hilbertian Generalized Autoregressive Conditional Heteroscedastic models and develop an appropriate calculus for their time domain analysis. Following recent advances in functional data theory and operatorial statistics, there are growing interests in processes with values in Hilbert spaces. Translating data into functions and then applying functional data techniques and models appear to be more effective than the classical data analysis in time series. We discuss existence and stationarity of Hilbertian Generalized Autoregressive Conditional Heteroscedastic models, and present a new method of parameter estimation, based on the principal components, in Functional Data Analysis. We also do a simulation procedure to generate values for certain Hilbertian Generalized Autoregressive Conditional Heteroscedastic models.

In the second part, we give a survey on Pettis conditional expectation of weak random elements in non-separable Banach spaces. We establish basic ingredients for the calculus of the Pettis conditional expectation of weak first-order scalarly measurable random elements with values in the dual space of a non-separable Banach space.We prove the continuity for the conditional expectation. We provide examples and calculating the conditional expectation of scalarly measurable but not

strongly measurable random elements with values in some non-separable Banach spaces.

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Chapter 1

Introduction

1.1 Aims

This thesis has two parts. In the first part (Chapter 1 and 2), we are going to introduce Hilbertian GARCH models. We will discuss existence and stationarity of these processes. We will present a new method of parameter estimation based on principal components in functional data models and simulate HGARCH process in the special case.

In the second part (Chapter 3), we are going to do a survey on Pettis conditional expectation of weak random elements and prove the convergence theorem for this conditional expectation. Finally, in two examples, we will calculate the conditional expectation of scalarly measurable, but not strongly measurable, random elements with values in some non-separable Banach spaces. Our idea to do this part of thesis, is to prepare necessary tools to define weak HGARCH models.
The main open problem for future work is to use Pettis conditional covariance to define weak Hilbertian GARCH models.

1.2 Preliminaries

In this chapter, we have a short review on the topics which will be used throughout this thesis. Some definitions and theorems which are needed in the next chapters are given. Chapter 2, is indeed the main chapter of this thesis. In this chapter, we will introduce the Hilbertian GARCH Models. In Chapter 3, the conditional expectation of *weak random elements* with values in non-separable Banach spaces will be introduced. We provide certain functional analysis concepts, univariate and multivariate GARCH models in Appendixes.

1.2.1 Notations

Let $\mathbb{Z}$,$\mathbb{C}$ and $\mathbb{R}$ stand for the set of integers, complex numbeers and real numbers, respectively. We denote $\overline{a}$ as the complex conjugate of a, and X, Y are used for Banach spaces with the norms $\|.\|_X$ and $\|.\|_Y$ respectively. We use $\mathbb{H}$ to show the Hilbert space with an inner product $\langle .,. \rangle_{\mathbb{H}}$ and the norm $\|.\|_{\mathbb{H}}$. Small Greek letters such as ξ, η and ζ are used for Banach (Hilbert)-valued random variables which are called random elements, we also use capital English letters such as A, B and C for operators. The notation A^* stands for the adjoint of operator A. The notation $\mathcal{B}(X)$ stands for the Borel σ-field, the smallest σ-field which is generated by open subsets of X. Also $\langle x^*, x\rangle$ is used to denote $x^*(x)$, when $x \in X$, $x^* \in X^*$, and X^* is the dual space of X. We use the notations E for expectation of real (complex)-valued random variables, $\mathbb{E}_{\mathrm{B}}$ for expectation of random elements in the sense of Bochner integral and $\mathbb{E}_{\mathrm{P}}$ for the expectation of scalarly measurable random elements in the sense of Pettis integral. The notation of the tensor product ,$\otimes$, will be extensively used in this thesis.

The notation $\mathbb{L}(X, Y)$ stands for the space of all bounded linear operators from X into Y, and we denote $\mathbb{L}(X, X)$ by $\mathbb{L}(X)$ whenever ther is no ambiguity. The symbols U_t, ϵ_t and Z_t are used for univariate stochastic processes and $\mathbf{U}_t$, $\boldsymbol{\epsilon}_t$ and $\mathbf{Z}_t$ are used for multivariate stochastic processes.

1.2.2 Tensor Products

In this part, we define the tensor product of two vectors, two matrices, two elements of a Hilbert space, two operators and two vector spaces.

1.2.2.1 The Tensor Product of two Vectors

The outer product $u \otimes v$ is equivalent to a matrix multiplication uv', provided that u is represented as a $m \times 1$ column vector and v as a $n \times 1$ column vector (which makes v' a row vector) where v' is the transpose of v. For instance, if $m = 4$ and $n = 3$, then

$$u \otimes v = uv' = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \\ u_4v_1 & u_4v_2 & u_4v_3 \end{bmatrix}.$$

1.2.2.2 The Tensor Product of Two Matrices

Let A and B be $k \times l$ and $m \times n$ matrices respectively. The tensor (Kronecker) product of matrices A and B is given by the matrix

$$A \otimes B = \begin{bmatrix} A_{11} & \cdots & A_{1l} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kl} \end{bmatrix} \otimes \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix} = \begin{bmatrix} A_{11}B & \cdots & A_{1l}B \\ \vdots & \ddots & \vdots \\ A_{k1}B & \cdots & A_{kl}B \end{bmatrix},$$

with the entires

$$A_{ij}B = \begin{bmatrix} A_{ij}B_{11} & \cdots & A_{ij}B_{1n} \\ \vdots & \ddots & \vdots \\ A_{ij}B_{m1} & \cdots & A_{ij}B_{mn} \end{bmatrix}.$$

1.2.2.3 The Tensor Product of two elements in a Hilbert space

The tensor product of two elements of a Hilbert space is defined as follows.
For $x, y, h \in \mathbb{H}$, $x \otimes y$ is an operator in $\mathbb{L}(\mathbb{H})$ which is defined by

$$(x \otimes y)h = \overline{\langle x, h \rangle}_{\mathbb{H}} y. \tag{1.1}$$

1.2.2.4 The Tensor Product of Two Hilbert Spaces

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two separable Hilbert spaces. We say the space $\mathbb{HS}(\mathbb{H}_1, \mathbb{H}_2)$ is the Hilbert tensor product of the separable Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$. We denote this space by

$$\mathbb{H}_1 \hat{\otimes} \mathbb{H}_2 = \mathbb{HS}(\mathbb{H}_1, \mathbb{H}_2),$$

and its elements by

$$M = \sum_{m,j=1}^{\infty} a_{mj} e_m \otimes f_j = \sum_{m=1}^{\infty} e_m \otimes y_m,$$

where

$$\{a_{mj}\}_{m,j} \in l^2(\mathbb{N} \times \mathbb{N}),$$

and

$$y_m = \sum_{j=1}^{\infty} a_{mj} f_j,$$

when the orthonormal bases $\{e_m\}$ and $\{f_j\}$ of $\mathbb{H}_1$ and $\mathbb{H}_2$ have been chosen and where $l^2(\mathbb{N} \times \mathbb{N})$ is the space of double indices square summable sequences.
We remark that the scalar product of $\mathbb{H}_1 \hat{\otimes} \mathbb{H}_2$ satisfies

$$\langle\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle\rangle = \langle x_1, x_2 \rangle_{\mathbb{H}_1} \langle y_1, y_2 \rangle_{\mathbb{H}_2}.$$

Here we state the theorem about the tensor product of Hilbert spaces.

Theorem 1.1 *Let $\mathbb{H}_1, \mathbb{H}_2$ and $\mathbb{H}_3$ be three separable Hilbert spaces. Then*

(i) $(\mathbb{H}_1 \hat{\otimes} \mathbb{H}_2) \hat{\otimes} \mathbb{H}_3$ *is isometric to* $\mathbb{H}_1 \hat{\otimes} (\mathbb{H}_2 \hat{\otimes} \mathbb{H}_3)$.

(ii) $\mathbb{H}_1 \hat{\otimes} \mathbb{R}$ *is isometric to* $\mathbb{H}_1$.

(iii) $\mathbb{H}_1 \hat{\otimes} \mathbb{H}_2$ *is isometric to* $\mathbb{H}_2 \hat{\otimes} \mathbb{H}_1$.

(iv) $(\mathbb{H}_1 \hat{\otimes} \mathbb{H}_2) \hat{\otimes} \mathbb{H}_3$ *is isometric to* $(\mathbb{H}_1 \hat{\otimes} \mathbb{H}_3) \times (\mathbb{H}_2 \hat{\otimes} \mathbb{H}_3)$.

For proof see [2] Chapter 12, Section 3, Proposition 1.

1.2.2.5 The Tensor Product of Two Linear Operators

Let $\mathbb{H}_i,\ i = 1, ..., 4$ be separable Hilbert spaces and consider the operators $A \in \mathbb{L}(\mathbb{H}_1, \mathbb{H}_2)$ and $B \in \mathbb{L}(\mathbb{H}_3, \mathbb{H}_4)$. We denote by $A \otimes B$ the continuous linear operator from $\mathbb{H}_1 \hat{\otimes} \mathbb{H}_3$ to $\mathbb{H}_2 \hat{\otimes} \mathbb{H}_4$, which is defined by $(A \otimes B)M = BMA^*$, where M is a linear operator from $\mathbb{H}_1$ into $\mathbb{H}_3$, $A \otimes B$ is called the tensor product of A and B. In the following theorem some properties of tensor product of linear operators are mentioned.

Theorem 1.2 *Let $A \in \mathbb{L}(\mathbb{H}_1, \mathbb{H}_2)$ and $B \in \mathbb{L}(\mathbb{H}_3, \mathbb{H}_4)$. The operator $A \otimes B \in \mathbb{L}(\mathbb{H}_1 \otimes \mathbb{H}_2), (\mathbb{H}_3 \otimes \mathbb{H}_4)$ has the following properties:*

(i)$\|A \otimes B\| \leq \|A\| \|B\|$.

(ii) $(A \otimes B)(x \otimes y) = Ax \otimes By$ *for each* $x \in \mathbb{H}_1$ *and* $y \in \mathbb{H}_3$.

(iii) $(A \otimes B)^* = A^* \otimes B^*$. *In particular, if A and B are both left (right)*

invertible, then $A \otimes B$ is left (right) invertible. If A and B are isomorphism then $A \otimes B$ is also isomorphism and the tensor product of two projectors is a projector too.

For proof see [2] Chapter 12, Sction 4, Proposition 1.

1.2.3 X-valued Random Variables

A pair $(\Omega, \mathcal{F})$, where Ω is a non-empty set and $\mathcal{F}$ is a σ-field of subsets of Ω, is called a measurable space and sets from $\mathcal{F}$ are called measurable sets. A measure is a real valued, non-negative, and countably additive set function μ, defined on $\mathcal{F}$ such that $\mu(\Phi) = 0$. A measure μ is called probability measure if $\mu(\Omega) = 1$ and usually is denoted by P. Also the triple $(\Omega, \mathcal{F}, P)$ is called the probability space.

Definition 1.1 *Let $(\Omega, \mathcal{F})$ and $(X, \mathcal{B}(X))$ be measurable spaces. A mapping $\xi : \Omega \longrightarrow X$ is said to be $\mathcal{F}/\mathcal{B}(X)$ measurable or strongly measurable if the inverse image $\xi^{-1}(B)$ lies in $\mathcal{F}$ for each $B \in \mathcal{B}(X)$.*

After defining measurable mappings, we can define X-valued random variables.

Definition 1.2 *Let $(\Omega, \mathcal{F})$ and $(X, \mathcal{B}(X))$ be measurable spaces. A measurable mapping $\xi : \Omega \longrightarrow X$ is called an X-valued random variable or a random element in $(X, \mathcal{B}(X))$. Also we shall say that ξ is a random element in X.*

From now on and throughout this thesis, we use the expression random elements for Banach (Hilbert)-valued random variables.

For random element ξ in X, $\langle x^*, \xi \rangle$ for each $x^* \in X^*$, is a complex or real-valued random variable and $\|\xi\|_X$ is a real valued random variable.

Another kind of measurability, used for random elements in this thesis, is the following.

Definition 1.3 *A random element ξ from Ω into X is called scalarly measurable if the complex-valued random variable $\langle x^*, \xi \rangle$ is measurable, i.e. $\mathcal{F}/\mathcal{B}(\mathbb{C})$ measurable for every $x^* \in X^*$.*

For random elements in Banach space X, we can introduce the notion of order of a random element in the following way.

Definition 1.4 *The random element $\xi : \Omega \longrightarrow X$ has the weak order p or it is of weak order p, $(0 < p < \infty)$, if $E\,|\langle x^*, \xi\rangle|^p < \infty$ for each $x^* \in X^*$, but if $E\,\|\xi\|^p < \infty$, we shall say that it has the strong order p or it is of strong order p, $(0 < p < \infty)$.*

It can be shown that the strong order implies the weak one and the converse implication is valid only if X is finite dimensional. See [8] and [77] for more on this subject.

Let $Ł^p(X, P)$ stand for the space of all strongly measurable random elements in X for which $E\|.\|_X^p < \infty$, $p > 0$; and $\mathcal{L}_w^p(X, P)$, $1 \leq p < \infty$, stands for the space of scalarly measurable weak random elements in X for which $\|.\|_p^w = \sup_{\|x^*\|\leq 1}(E|\langle x^*, .\rangle|^p)^{1/p}$ is finite. Also let $\mathcal{L}_{w*}^\infty(X^*, \mu)$ denote the space of scalarly measurable random elements ζ in X^* equipped with the norm

$$\|\zeta\|_\infty^{w^*} = \sup_{\|x\|\leq 1} ess\sup_{\omega\in\Omega} |\langle \zeta, x\rangle|, x \in X, \zeta \in X^*.$$

In the next definition, we define independency for random elements in X and $\mathbb{L}(X)$.

Definition 1.5 *Two random elements ξ and η are said to be independent if $\langle \xi, x^*\rangle$ and $\langle \eta, y^*\rangle$ are independent for all $x^*, y^* \in X^*$; the independence is similarly defined for finite or infinite families of random elements. Two random elements A and B in $\mathbb{L}(X)$ are said to be independent if for all x and y in X, Ax and By are independent random elements in X.*

1.2.3.1 Expectation

In this subsection, we define two kinds of expectations for random elements.

Definition 1.6 *A measurable function $\xi : \Omega \to X$ is called Bochner integrable if there exists a sequence of simple functions ξ_n such that*

$$lim_{n\to\infty} \int_\Omega \|\xi_n - \xi\| \,\mathrm{d}\mu = 0.$$

In this case we denote the Bochner integral of ξ *over the set* E *with respect to* μ *by* $(\mathrm{B})\text{–}\int_E \xi \, \mathrm{d}\mu$ *which is defined for each* $E \in \mathcal{F}$ *by*

$$(\mathrm{B})\text{–}\int_E \xi \, \mathrm{d}\mu = lim_{n\to\infty} \int_E \xi_n \, \mathrm{d}\mu$$

where $\int_E \xi_n \, \mathrm{d}\mu$ *is defined in the obvious way.*
If ξ *is strongly measurable and if* $\int \|\xi\| \, \mathrm{d}\mu < \infty$ *then the Bochner integral of* ξ *exists as an element of* X*. When* $\mu(\Omega) = 1$*- i.e. when* μ *is probability measure- then* $\int\limits_{\Omega} \xi \, \mathrm{d}\mu$ *is also called the expectation of* ξ *and is denoted by* $\mathbb{E}_{\mathrm{B}}\xi$*.*

As seen in the Definition 1.6, $\mathbb{E}_{\mathrm{B}}\xi$ exists only for strongly measurable random elements. When the scalarly measurable weak random elements are used, we present the following definition for the expectation.

Definition 1.7 *A random element* $\xi : \Omega \to X$ *is called Pettis integrable with respect to* μ*, if*

(i) $\xi \in \mathcal{L}^1_w(X, \mu)$*,*

(ii) For every $E \in \mathcal{F}$*, there exists an element* ξ_E *in* X *such that,*

$$\langle x^*, \xi_E \rangle = \int_E \langle x^*, \xi \rangle \, \mathrm{d}\mu, \quad \textit{for every } x^* \in X^*. \tag{1.2}$$

The element ξ_E *is called the Pettis integral of* ξ *over* E *with respect to the measure* μ *and it is denoted by* $(\mathrm{P})\text{–}\int_E \xi \, \mathrm{d}\mu$*. In particular, when* μ *is a probability measure,* ξ_Ω *stands for the expectation of* ξ *and is denoted by* $\mathbb{E}_{\mathrm{P}}\xi$*.*

If X is a reflexive Banach space, then every separably-valued random element of weak order one is Pettis integrable, [77].

1.2.3.2 Conditional Expectation

In this subsection, we introduce conditional expectation given a σ -field for strong and weak order random elements based on suitable integrations in Banach spaces.

Definition 1.8 *For* $\xi \in Ł^1(X, P)$ *and given* $\mathcal{G}$*, a sub-*σ*-field in* $\mathcal{F}$*, the conditional expectation of* ξ *given* $\mathcal{G}$ *is defined to be a random element in* X*, denoted by* $\mathbb{E}_{\mathrm{B}}[\xi|\mathcal{G}]$*, which is measurable* $\mathcal{G}$ *and satisfies the equation*

$$(\mathrm{B})\text{–}\int_A \xi \, \mathrm{d}P = (\mathrm{B})\text{–}\int_A \mathbb{E}_{\mathrm{B}}[\xi|\mathcal{G}] \, \mathrm{d}P, \textit{ for all } A \in \mathcal{G}, \tag{1.3}$$