IN THE NAME OF GOD

THE ZARISKI TOPOLOGY ON THE PRIME SPECTRUM OF A MODULE

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dedicated:

To my dear parents

and

To whom I Love

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ABSTRACT

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Let M be a module over a commutative ring R. A submodule K of M is called prime if $K \neq M$ and whenever $r \in R$ and $m \in M$ satisfy $rm \in K$ then $r \in (K : M)$ or $m \in M$, where $(K : M) = \{r \in R : rM \subseteq K\}$. Clearly this is a generalization of the notion of prime ideals of rings.

The prime spectrum $spec(M)$ of M is the collection of all prime submodules. We topologize $spec(M)$ with the Zariski topology, which is analogous to that on $spec(R)$. Then define continuous map ψ from $spec(M)$ to $spec(\bar{R})$ (where $\bar{R} = \frac{R}{Ann(M)}$) find condition that ψ is surjective, open, closed, injective and homeomorphic.

We find base for Zariski topolog on $spec(M)$, and prove this base and $spec(M)$ are quasi-compact.

We find subsets, Y of $spec(M)$ that are irreducible, irreducible closed and generic point for $spec(M)$ and every irreducible closed subset of $spec(M)$.

We prove that $spec(M)$ is T_0-space iff is injective and find condition under which $spec(M)$ be a T_1 and spectral space.

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CHAPTER I
INTRODUCTION

0. Literature Survey

M. Hochster has characterized spectral spaces as quasi-compact T_0-spaces W such that W has a quasi-compact open base closed under finite intersections and each irreducible closed subset of W has a generic point. We follow the Hochster's characterization closely in discussing whether $spec(M)$ of a module M is a spectral space. [6]

In 1984 Chin-pi-Lu wrote a paper about prime submodules of modules. The purpose of this paper is to introduce interesting and useful properties of prime submodules of modules and show various applications of the properties. [9]

In 1988 Z. A. El-Bast and P. F. Smith discussed the multiplication modules. [5]

In 1989 C. P. Lu discussed M-radicals of submodules, the M-radical of a submodule N in a module M over a ring R is defined as the intersection of all prime submodules. [10]

In 1992 J. Jenkins and P. F. Smith discussed, the prime radical of a module over a commutative ring. In this paper they proved that for every

Dedekind domain R, an R-module M, the radical of M has a certain form. [8]

In 1992 R. L. McCasland and M. E. Moore discussed about prime submodules. According to this paper in many cases the conclusions about finitely generated modules over a PID are shown to be valid for modules including infinitely generated ones, over an arbitrary integral domain. [12]

In 1993 R. L. McCasland and P. F. Smith discussed prime submodules of Noetherian modules. [13]

In 1994 T. Duraviel discussed a topology on spectrum of modules, that he defines a topology on spectrum of R-modules by means of its prime submodules and proves some results which are already known for $spec(R)$. And defines absolutely flat R-modules which is a generalization of an absolutely flat ring and prove somes related results. [4]

In 1995 C. P. Lu discussed spectra of modules. The spectra of modules are introduced and a useful relationship between $spec(M)$ and $spec(M_s)$ are gained. [11]

In 1997 R. L. McCasland, M. E. Moore and P. F. Smith discussed on the spectrum of a module over a commutative ring. This paper investigates when the spectrum of M consisting of all prime submodules of M has a Zariski topology analogous to that for R. For finitely generated modules M this occurs if and only if M is a multiplication module. [14]

1. The Scope of the Dissertation

Throught this dissertation, all rings R are commutative with identity and all modules are assumed to be unitary.

Let R be a ring and let M be an R-module. A submodule K of M is called prime if $K \neq M$ and whenever $r \in R$ and $m \in M$ satisfy $rm \in K$ then $r \in (K : M)$ or $m \in K$, where

$$(K : M) = \{r \in R : \; rM \subseteq K\}.$$

For any module M over a commutative ring R with identity the prime spectrum $spec(M)$ of M is the collection of all prime submodules. In section 2 of this chapter we study some properties of prime submodules of a module. Then in section 3 we define the Zariski topology on $spec(R)$ and extend this notion to modules and define Zariski topology and quasi- Zariski topology on $spec(M)$, then we study some relationship between $spec(M) = X$ and $spec(\frac{R}{Ann(M)}) = X^{\bar{R}}$.

In chapter II we define the natural map $\psi : spec(M) \longrightarrow spec(\frac{R}{Ann(M)})$ by $\psi(P) = \overline{(P : M)}$ and find conditions under which ψ is injective, surjective, open, closed, or homeomorphic. We also consider some relationship between X and $X^{\bar{R}}$ with respect to connectedness. In section 2 of this chapter we introduce quasi-compact base for $spec(M)$.

In chapter III we find subsets, Y of $X = spce(M)$ that are closed, irreducible, and condition under which Y is irreducible closed subset of X, then we find generic point for $spec(M)$ and every irreducible closed subset of $spec(M)$.

M. Hochster [6] has characterized spectral spaces as quasi-compact T_0-spaces W such that W has a quasi-compact open base closed under finite intersections and each irreducible subset of W has a generic point. In chapter IV we follow the Hochster's characterization closed in discussing whether $spec(M)$ of a module M is a spectral space. The injectivity and the surjectivity of the natrual map ψ of X have important roles for X being spectral. We prove that X is T_0-space iff ψ is injective iff X has at most one p-prime submodule for every $p \in spec(R)$. We show that if M is a finitely generated nonzero R-module, then X is a spectral space iff M is a multiplication module iff X is homeomorphic to $spec(\frac{R}{Ann(M)})$ iff ψ is injective. We also consider the following two cases:

a) The image of ψ is a closed subset of $spec(\frac{R}{Ann(M)})$, and

b) X is a non-empty finite set.

For each of these cases, we prove that X is a spectral space iff ψ is injective.

2. Prime Submodules

Let M be an R-module. For any submodule N of M we denote the annihilator of $\frac{M}{N}$ by $(N : M)$, i.e.

$$(N : M) = \{r \in R : \; rM \subseteq N\}.$$

Definition 1.2.1. Let R be a ring and let M be an R-module. A submodule K of M is called prime if $K \neq M$ and whenever $r \in R$ and $m \in M$ satisfy $rm \in K$ then $r \in (K : M)$ or $m \in K$.

Clearly, any prime ideal of R is a prime R-submodule of the R-module R.

Example 1.2.2. The torsion submodule $T(M)$ of M over an integral domain is a prime submodule if $T(M) \neq M$ because if $rm \in T(M)$ for some $0 \neq r \in R$ and some $m \in M$, then there exists $0 \neq r' \in R$ such that $r'rm = 0$. Since R is a domain, $rr' \neq 0$ and so $m \in T(M)$. Clearly, if $r = 0$ then $r \in (T(M) : M)$.

Lemma 1.2.3. A submodule K of an R-module M is prime if and only if $P = (K : M)$ is a prime ideal of R and the $(\frac{R}{P})$-module $\frac{M}{K}$ is torsion free.

Proof. Let K be a prime submodule of M. Also suppose that $rr' \in P$ and $r \notin P$ for some $r, r' \in R$. Then $rr'M \subseteq K$ and since $r \notin P$, $r'M \subseteq K$. Thus $r' \in P$ and P is a prime ideal of R. Now we know that $\frac{M}{K}$ is an $\frac{R}{P}$-module, because $P = Ann(\frac{M}{K})$. Now suppose that $(r + P)(m + K) = K$ for some $r + P \in \frac{R}{P}$ and $m + K \in \frac{M}{K}$ therefore $rm + K = K$ and hence $rm \in K$. Consequently $r \in P$ or $m \in K$ i.e. $r + P = P$ or $m + K = K$. Thus the $\frac{R}{P}$-module $\frac{M}{K}$ is torsion-free. Conversely, we assume that $rm \in K$ and $r \notin P$, where $r \in R$ and $m \in M$. Hence $rm + K = (r + P)(m + K) = K$. Since $\frac{M}{K}$ is a torsion-free $\frac{R}{P}$-module then $m + K = K$ and so $m \in K$. It follows that K is a prime submodule of M. □

If K is a prime submodule of M and $p = (K : M)$ then K is called a p-prime submodule of M.

Example 1.2.4. If R is a simple ring, then every non-zero R-module M of R is torsion-free, since for any $0 \neq x \in M$, $ann(x) \neq R$ and hence $Ann(x) = 0$. Also for any proper submodule N of M, $(N : M) = 0$ and since (0) is the only maximal ideal of R, (0) is prime. It follows that a simple ring R has the property that every proper submodule N of M is prime. $\square$

Corollary 1.2.5. Let K be any submodule of an R-module M such that $(K : M)$ is a maximal ideal of R. Then K is a prime submodule of M. In particular, mM is a prime submodule of an R-module M for every maximal ideal m of R such that $mM \neq M$.

Proof. Since $(K : M) \neq R$ then $K \neq M$ and since $(K : M) = m$ is a maximal ideal of R then $\frac{R}{m}$ is a field and $\frac{M}{K}$ is a vector space over $\frac{R}{m}$. Now if $\bar{r}\bar{x} = 0$ and $\bar{r} \neq 0$, where $\bar{r} = r + M$ for some $r \in R$ and $\bar{x} = x + K$ for some $x \in M$, then $\overline{r^{-1}}\bar{r}\bar{x} = 0$ and so $\bar{x} = 0$. Thus $\frac{M}{K}$ is torsion-free $\frac{R}{m}$-module. It follows that K is a prime submodule of M by Lemma 1.2.3. Now if for some maximal m of R $mM \neq M$ then it is clear that $(mM : M) = m$. Thus mM is a prime submodule of M. $\square$

Example 1.2.6. Every proper subspace of a vector space is prime.

Proof. Let V be a vector space over the field F and W be a proper subspace of V. Since $rV = V$ for every $0 \neq r \in F$ then $(W : V) = 0$ and since $< 0 >$ is a maximal ideal of F therefore by Corollary 1.2.5 W is a prime submodule of V.

Corollary 1.2.7. Let N be a proper submodule of an R-module M and let m be a maximal ideal of R. Then N is m-prime if and only if $mM \subseteq N$. Consequently, if N is an m-prime submodule of M, then so is every proper submodule of M containing N.

Proof. The necessity is trivial. Conversely if $mM \subseteq N$ then $m \subseteq (N : M)$ and since $N \neq M$ hence $(N : M) \neq R$ therefore $m = (N : M)$. It follows that N is an m-prime submodule of M by Corollary 1.2.5. $\square$

Proposition 1.2.8. If N is a maximal submodule of an R-module M, then $(N : M)$ is a maximal ideal of R and N is a prime submodule of M.

Proof. Let $(N : M) \subseteq m \subseteq R$, where m is an ideal of R. Since N is a maximal submodule of M, hence $\frac{M}{N}$ is a simple R-module. It implies that $\frac{M}{N}$ is cyclic and $\frac{M}{N} = (x+N)R$ for some $x \in M$. Thus $m(\frac{M}{N}) = \frac{M}{N}$ or $m(\frac{M}{N}) = 0$. If $m(\frac{M}{N}) = \frac{M}{N}$ then $m(\frac{M}{N}) = (x + N)R$ and hence there exists $r_i \in m$ and $y_i + N \in \frac{M}{N}$ $(y_i \in M)$ such that $x + N = \sum_{i=1}^{n} r_i(\sum_{i=1}^{t} y_i + N)$. On the other hand, $y_i + N = \sum_{i=1}^{t} r'_i(x + N)$, for some $r' \in R$, therefore

$$(x - (\sum_{i=1}^{n} r_i)(\sum_{j=1}^{t} r'_j)x) + N = (1 - (\sum_{i=1}^{n} r_i)(\sum_{i=1}^{t} r'_j))(x + N) = 0.$$

It follows that $1 - (\sum r_i)(\sum r'_j) \in Ann(\frac{M}{N}) = (N : M) \subseteq m$. Since $\sum_{i=1}^{n} r_i \in m$, $1 = 1 - (\sum r_i)\sum r'_j + \sum r_i \sum r'_j \in m$ so $m = R$. Now if $m(\frac{M}{N}) = 0$, then $mM \subseteq N$ and so $m \subseteq (N : M) \subseteq m$. Hence $(N : M) = m$. Therefore $(N : M)$ is a maximal ideal of R. By Corollary 1.2.5 N is a prime-submodule of M. $\square$

Remark 1.2.9. If m is a maximal ideal of a ring R, then not every m-prime submodule of an R-module M is a maximal submodule. In Example 1.2.6 we can see that $< 0 >$ is a maximal ideal and all maximal or non-maximal subspaces of vector space V are $< 0 >$-prime submodules in V.

Corollary 1.2.10. If M is a finitely generated module, then every proper submodule of M is contained in a prime submodule.

Proof. Let N be a proper submodule of M and let A be the set of all submodules of M containing N. A is non-empty, because $N \in A$. By Zorn's

Lemma, it can easily be proved that there exists a maximal element L in A. Thus L is a maximal submodule of M and by Proposition 1.2.8 L is a prime submodule of M containing N. $\square$

Definition 1.2.11. An R-module M is called a multiplication module provided that for every submodule N of M there exists an ideal I of R such that $N = IM$.

Theorem 1.2.12. Let M be a non-zero R-module, where $R \neq 0$. If M is a multiplication module, then M has at least one prime submodule.

Proof. Let $M \neq 0$ and $0 \neq m \in M$. Then $I = \{r \in R|\ rm = 0\}$ is a proper ideal of R and hence $I \subseteq P$ for some maximal ideal P of R. If $M = PM$ then since $Rm = AM$, for some ideal A of R, we have $Rm = AM = PAM = PRm = Pm$. Therefore $(1-r)m = 0$ for some $r \in P$ and hence $(1-r) \in I$. Since $I \subseteq P$ then $(1-r) \in P$ and so $1 \in P$, a contradication. Thus $M \neq PM$. Since $(PM : M) = P$ is a maximal ideal, PM is a prime submodule of M by Corollary 1.2.5. $\square$

For any R-module M, let $spec(M)$ denotes the colleciton of all prime submodules of M. Now let H be any R-module, for any prime ideal p of R we define

$$spec_p(H) = \{L \in spec(H)|\ (L : H) = p\}.$$

Lemma 1.2.13. Let p be a prime ideal of R and let M be an R-module. Let N be any submodule of M and let $K \in spec_p(M)$, then $K \cap N = N$ or $K \cap N \in spec_p(N)$.

Proof. Let $K \cap N \neq N$ for any $r \in p$ we have $rN \subseteq rM \subseteq K$, also $rN \subseteq N$ then $rN \subseteq K \cap N$. Hence $p \subseteq (K \cap N : N)$. Now suppose that $r \in (K \cap N : N)$ then $rN \subseteq K \cap N \subseteq K$. Since $N \nsubseteq K$ and K is a prime submodule of M then $r \in p$. Thus $(K \cap N : N) = p$. Let $rx \in K \cap N$, where $r \in R$ and $x \in N$, hence $rx \in K$ and so $r \in p$ or $x \in K$. It follows that $r \in p$ or $x \in K \cap N$. Thus $K \cap N \in spec_p(N)$.

3. Zariski Topology on $spec(M)$

Recall that $spec(R)$ denotes the collection of all prime ideals of R. For an ideal I of R we define

$$V(I) = \{P \in spec(R) : \ I \subseteq P\}.$$

It can easily be checked that $V(\{0\}) = spec(R)$ also

$V(R) = \emptyset$

$V(I) \cup V(J) = V(IJ)$

$\bigcap_{\lambda \in \Lambda} V(I_\lambda) = V(\sum_{\lambda \in \Lambda} I_\lambda)$

where I and J and I_λ $(\lambda \in \Lambda)$ are ideals of R. Thus the $V(I)$ are the closed sets for a topology on $spec(R)$, called the Zariski topology.

Now we extend this notion to modules. For any submodule N of an R-module M we define $V(N)$ to be set of all prime submodules of M containing N. Of course $V(M)$ is just the empty set and $V(0)$ is $spec(M)$.

Let I be an ideal of a ring R. Define the variety of I denoted by $V^R(I) = V(I)$. The collection $\varsigma(R) = \{V^R(I) | \ I \trianglelefteq R\}$ of all varieties of ideals I of R satisfies the axioms for closed sets in a topological space.

Now let M be an R-module for any submodule N of M we consider two different types of varieties denoted, by $V^*(N)$ and $V(N)$ respectively as follows:

$$V^*(N) = \{P \in spec(M) \mid P \supseteq N\}.$$

Then

(i) $V^*(0) = spec(M)$ and $V^*(M) = \emptyset$

(ii) $\bigcap_{i\in\Lambda} V^*(N_i) = V^*(\sum_{i\in\Lambda} N_i)$ for any index set Λ

(iii) $V^*(N) \cup V^*(L) \subseteq V^*(N \cap L)$, where $N, L, N_i \leq M$ since

$$\left.\begin{array}{l} N \cap L \subseteq N \Rightarrow V^*(N \cap L) \supseteq V^*(N) \\ N \cap L \subseteq L \Rightarrow V^*(N \cap L) \supset V^*(L) \end{array}\right\} \Rightarrow V^*(N) \cup V^*(L) \subseteq V^*(N \cap L).$$

We denote the set $\{V^*(N) \mid \ N \leq M\}$ by $\varsigma^*(M)$.

Next, we define

$$V(N) = \{P \in spec(M) \mid \ (P : M) \supseteq (N : M)\},$$

which are the closed sets. Then

a) $V(0) = spec(M)$ and $V(M) = \emptyset$

b) $\bigcap_{i\in\Lambda} V(N_i) = V(\sum_{i\in\Lambda} (N_i : M)M)$

c) $V(N) \cup V(L) = V(N \cap L)$ where $N, L, N \leq M$.

Proof. (a) The proof is trivial.

(b)

$$\begin{aligned} P \in \bigcap_{i\in\Lambda} V(N_i) &\iff (P : M) \supseteq (N_i : M) \qquad \forall i \in \Lambda \\ &\iff (P : M)M \supseteq (N_i : M)M \qquad \forall i \in \Lambda \end{aligned}$$

since $((N : M)M : M) = (N : M)$ for every submodule N of M because

$$\text{if} \quad r \in (N : M)$$