بسم الله الرحمن الرحیم

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Faculty of Mathematical Sciences

Department of Mathematics

Ph.D. Thesis

Bounded Matrices and Frame Theory

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January 2014

Vali-e-Asr University of Rafsanjan
Faculty of Mathematical Science
Department of Mathematics

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A Dissertation Submitted as a Partial Fulfillment of the Requirement for the Degree of Doctor of Philosophy (Ph.D.) in Pure Mathematics

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Acknowledgement

All praise is due to Allah, the Lord of the Worlds, and peace and greetings be upon our Master and Prophet, Ab-al Qassem al-Mustafa Muhammad and upon his immaculate, pure and chosen household, especially the one remaining with Allah on earth, Vali-e-Asr, Imam Mahdi (a.s.). Again, I thank God Almighty who gives me the opportunity of benefiting of the outstanding moral and scientific character of my supervisor professor Mohammad Ali Dehghan whose creative and original thinking on frames and other areas of mathematics have been a constant source of inspiration for me. I can still vividly recall how he cleared my despair on reading the long papers on frames by several penetrating words. Furthermore, he elaborates me how a person is having great zeal for doing and enjoying research. I send my loving thanks for my parents, who supported me amorously. I would like also to express my gratefulness to my dear wife. Without her understanding it would have been imposible for me to finish this work. I am also grateful to Professor Billy Rhoades, Professor Grahame Bennett, Professor Graham Jameson, Professor Bilal Altay, Dr. Ali Armandnejad and Dr. Hamidreza Afshin for their attentions and answering my questions during the recent years . Thanks also goes to all honorable referees: Professor M. Radjabalipour, Professor R. A. Kamyabi Gol, Dr. Foroutannia and Dr. Safapour for their valuable comments and suggestions which improved this dissertation. Finally, I would like to record my pleasure to Shahid Sayyid Gholamreza Mirafzali and Shahid Sayyid Mohammad Reza Mirafzali for their Spiritual support.

Dedicated to:

Hazrat Fatima Zahra (as) ,

Leader of the Women of Paradise

السّلام علیک یا فاطمة الزهراء (س)

ما گوشه نشینان غم فاطمه ایم　　محتاج عطا و کرم فاطمه ایم

عمریست که از داغ غمش سوخته ایم　　دلسوخته عمر کم فاطمه ایم

Abstract

One of the aims of this dissertation is the consideration of the boundedness problem of infinite matrix operators on some sequence spaces such as the Euler weighted sequence space $e^{\theta}_{w,p}$, the $p-$bounded variation sequence space bv_p, the sequential *weak* ℓ_p space and the block weighted sequence space $\ell_p(w, F)$. Our investigations consider some non-negative infinite matrices such as Nörlund matrices, generalized Hausdorff matrices, Weighted mean matrices and more generally the lower triangular matrices as operators on the mentioned spaces and present lower bound and upper bound for them. Our results generalize some work of Bennett, Rhoades, Chen, Jameson and Foroutannia and also my earlier results joint with Lashkaripour.

Another purpose of this dissertation is to introduce the concept of E$-$frames in a separable Hilbert space $\mathcal{H}$, where E is an invertible infinite matrix mapping on the Hilbert space direct sum $\overset{\infty}{\underset{n=1}{\oplus}} \mathcal{H}$. We investigate and study some properties of E$-$frames and characterize all E$-$frames in $\mathcal{H}$. Further more, we characterize all dual E$-$frames associated with a given E$-$frame. A similar characterization is also established for E$-$orthonormal bases, E$-$Reisz bases and dual E$-$Riesz bases. In continue we deal with several special types of E$-$frames such as $\boldsymbol{\Delta}-$frames and Euler frames for $\mathcal{H}$ which are related to the spaces bv_p and the Euler weighted sequence space $e^{\theta}_{w,p}$, respectively. Our results generalize the concept of frames because the ordinary frames are the special case of E$-$frames in which the matrix E be replaced by the identity matrix operator I on $\overset{\infty}{\underset{n=1}{\oplus}} \mathcal{H}$.

Keywords: Sequence space, Bounded matrix, Lower bound; Norm; Frame.

Introduction

The boundedness problem of linear operators on normed spaces have been considered from the distant past. The results about norm and upper bound of matrix operators on sequence spaces went right back to the original theorems of Hardy, Copson and Hilbert which are summarized together in the famous book of Hardy, Littlewood and Polya [25]. But the lower bound problem for matrix operators on sequence spaces have been started by Lyons [37] in 1982 via computing the lower bound of Cesàro matrix on the sequence space ℓ_2. The technique used by Lyons was very complicated and did not generalize to other matrices.

In 1986, Bennett [3] began working on this problem and came up with an explicit formula for the solution for lower bound of any matrix with non-negative entries which is a bounded operator on an ℓ_p space for $1 \leq p \leq \infty$. Unfortunately, using this formula to compute the lower bound for specific classes of matrices is often very difficult. In spite of this fact, Bennett in [4] found the lower bound for the calss of the Hausdorff matrices, that are bounded operators on ℓ_p for some $1 \leq p < \infty$. He also computed the value of lower bound for quasi-Hasudorff matrices. Moreover in [5], he obtained a Hardy type formula for the lower bound of Hausdorff matrices on the sequence space $\ell_p(0 < p < 1)$.

In 1996, Johnson, Mohapatra and Rass in [29] obtained an upper estimate and a lower estimate for the norm of Nörlund matrices on space ℓ_p where $1 \leq p < \infty$.

In 1999, Jameson [27] considerd the analogous problem in the continuous case on the Lorentz sequence space $d(w, 1)$. Afterwards, Jameson, Lashkaripour

[28] and Foroutannia [31] generalized this results to the weighted sequence space $\ell_p(w)$, the Lorentz sequence space $d(w,p)$ and the block weighted sequence space $\ell_p(w,F)$ where $1 \leq p < \infty$.

Between years 2008 to 2011, Chen and Wang [10, 11, 12, 13] generalized some results of Bennett about the lower bound of matrices to the sequence space ℓ_p where $0 < p < 1$ or $-\infty < p < 0$. For example, they obtained a general lower estimate and upper estimate for the exact velue of the lower bound of lower triangular non-negative matrices, and obtained a Hardy type formula for the lower bound of generalized Hausdorff matrices.

In 2011, Lashkaripour and Talebi [32, 33, 34, 35] generalized this results to the weighted sequence space $\ell_p(w)$, block weighted sequence space $\ell_p(w,F)$ and the Euler weighted sequence space $e^{\theta}_{w,p}$ where $0 < p < 1$ or $-\infty < p < 0$.

As a part of this thesis, these results are extended to the matrix operators on the $p-$bounded variation sequence space bv_p and the sequential *weak* ℓ_p where $0 < p < 1$, and on the Euler weighted sequence space $e^{\theta}_{w,p}$ where $1 \leq p < \infty$ and also on the block weighted sequence space $\ell_p(w,F)$ where $0 < p < 1$ or $-\infty < p < 0$. Another purpose of this thesis is to correlate the concept of frame in a separable Hilbert space $\mathcal{H}$ to the concept of boundedness of matrix operators on two sequence space $(bv_p, ||.||_{bv_p})$ and $(e^{\theta}_{w,p}, ||.||_{e^{\theta}_{w,p}})$.

A frame in $\mathcal{H}$, that historically was introduced by Duffin and Schaeffer [18] in 1952, is a countable family $\{f_k\}$ in $\mathcal{H}$ for which there exist positive real numbers A and B such that

$$A\|f\|^2 \leq \|\{\langle f, f_k\rangle\}_{k=1}^{\infty}\|_{\ell_2}^2 \leq B\|f\|^2, \ \forall f \in \mathcal{H}. \tag{1}$$

Frames have very important and interesting properties which make them very useful in the characterization of function spaces, signal processing and many other fields such as image processing, data compressing, sampling theory and so on. The work of Duffin and Schaeffer was not continued until 1986 when Daubechies, Grossmann and Meyer [16] applied the theory of frame to wavelet and Gabor transform. Theory of frames for Hilbert spaces have been developed deeply. Many people have done momentous works in this field such as Han, Larson, Young, Casazza, Christensen and Cao (see [7, 9, 14, 23, 43]).

In 1990, Grochenig, Aldroubi, Sung and Tang began to study the theory of frames for Banach spaces. They introduced two kinds of notions of frames for Banach spaces: Banach frames and $p-$frames $(1 < p < \infty)$. A sequence $\{g_k\}$ in the dual space X^* of a Banach space X is a $\mathbf{p}-$frame in X if there exist positive real numbers A and B such that

$$A\|x\| \leq \|\{\langle x, g_k\rangle\}_{k=1}^{\infty}\|_{\ell_p} \leq B\|x\|, \quad \forall x \in X. \tag{2}$$

A Banach frame with respect to ℓ_p for X is a $p-$frame in X with a reconstruction operator S (see[1, 21]). Subsequently, Casazza, Christensen and Stoeva in [8] generalized the concept of $p-$frames and introduce $\mathbf{X_d}-$frames, where X_d is a $BK-$space. In [36], Li and Cao introduced the notion of X_d frames and X_d Riesz bases for Banach spaces.

Keeping in mind the inequalities (1) and (2), working on the boundedness of matrices on the two sequence spaces $(bv_p, ||.||_{bv_p})$ and $(e^{\theta}_{w,p}, ||.||_{e^{\theta}_{w,p}})$, we became interested in studying the sequences $\{f_k\}$ in a Hilbert space $\mathcal{H}$, for which there exist positive real numbers A and B such that

$$A\,\|f\| \leq \|\{\langle f, f_k\rangle\}_{k=1}^{\infty}\|_{bv_2} \leq B\,\|f\|\,, \quad \forall f \in H \tag{3}$$

or

$$A\,\|f\| \leq \|\{\langle f, f_k\rangle\}_{k=1}^{\infty}\|_{e^{\theta}_2} \leq B\,\|f\|\,, \qquad \forall f \in \mathcal{H}. \tag{4}$$

For some reasons we named the first ones $\mathbf{\Delta}-$frames and the second ones Euler frames. This two types of frames motivated us to introduce and study a more general concept of frames namely E$-$frames for a separable Hilbert space $\mathcal{H}$, where

E is an invertible infinite matrix mapping on the Hilbert space direct sum $\overset{\infty}{\underset{n=1}{\oplus}} \mathcal{H}$. This is the second purpose of this thesis.

This Thesis is organized as follows. Chapter 1, contains some preliminaries which are essential to our discussions in the subsequent chapters. Some elementary concept of functional analysis such as boundedness, linearity of operators on normed spaces and certain sequence spaces and some preliminaries in frame theory such as frames, Riesz bases, orthonormal bases in Hilbert spaces are another concepts that are listed in chapter 1.

In chapter 2, we consider the Hausdorff matrices as operators from the weighted sequence space $\ell_p(w)$ into the Euler weighted sequence space $e^{\theta}_{w,p}(1 < p < \infty)$ where $0 < \theta < 1$ and obtain an upper estimate for their operator norm. Then we apply our results to some famous classes of Hausdorff matrices such as Cesàro matrices, Hölder matrices, Euler matrices and Gamma matrices. Our result generalizes Theorem 2.2 of [30]. Also we establish a similar results for the Nörlund matrix as an operator from $\ell_p(w)$ into the space $e^{\theta}_{w,p}$.

In chapter 3, we consider the generaized Hasdorff matrix operators as matrix mappings from the weighted sequence space $\ell_p(w)$ into either the sequential *weak* ℓ_p space or the block weighted sequence space $\ell_p(w, F)$, and obtain a lower estimate for their lower bound. A similar result is also established for their transpose. As a consequence, we apply our results to some famous classes of generaized Hausdorff matrices such as generaized Cesàro matrices, generaized Hölder matrices and generaized Gamma matrices. Our results generalize some results in [13] and [32]. Also in this chapter we consider the transpose of a lower triangular matrix with non-negative entries and increasing rows as a matrix selfmap of the $p-$bounded variation sequence space bv_p and establish a lower estimate for its lower bound. Moreover, we consider the transpose of an arbitrary lower triangular matrix with non-negative entries as a matrix mapping from either the weighted sequence space $\ell_p(w)$ into the sequential *weak* ℓ_p space or from the Euler weighted sequence space $\ell_p(w, I)$ into the block weighted sequence space $\ell_p(w, F)$ and discuss about its lower bound. Our results generalize some work in [12], [32], [33] and [35].

In chapter 4, we introduce the concepts of E$-$frames, E$-$Riesz bases and E$-$orthonormal bases for an arbitrary separable Hilbert space $\mathcal{H}$, where E is an invertible infinite matrix mapping on the Hilbert space $\overset{\infty}{\underset{n=1}{\oplus}} \mathcal{H}$. We study some of their properties and characterize all of them starting with an arbitrary orthonormal basis in $\mathcal{H}$. Further more, we characterize all dual E$-$frames associated with a given E$-$frame. Our results generalize the concept of frames because the ordinary frames are a special case of E$-$frames in which the matrix E is replaced by the identity matrix operator I on $\overset{\infty}{\underset{n=1}{\oplus}} \mathcal{H}$. Also in this section, we study the concept of $\boldsymbol{\Delta}-$frames and Euler frames as special cases of E$-$frames for $\mathcal{H}$ which are related to the Hilbert spaces bv_2 and e_2^θ, respectively. We compare this two type of frames with the ordinary frames, and characterize all $\boldsymbol{\Delta}-$frames, $\boldsymbol{\Delta}-$Riesz bases and $\boldsymbol{\Delta}-$orthonormal bases starting with an arbitrary orthonormal basis for $\mathcal{H}$. A similar characterization is also presented for all Euler frames, Euler Riesz bases and Euler orthonormal bases in $\mathcal{H}$. Moreover, in this section all dual $\boldsymbol{\Delta}-$frames and all dual Euler frames associated with a given $\boldsymbol{\Delta}-$frame and Euler frame, respectively, are identified. Finally, a similar result is also obtained for a new special type of E$-$frames namely Hausdorff frames.

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Chapter 1

Basic consepts and preliminaries results

In this chapter, we introduce some basic concepts which are essential to our discussions in the subsequent chapters. Some elementary concepts of functional analysis such as boundedness and linearity of operators on normed spaces and certain sequence spaces are studied in Section 1.1. Also, some preliminaries in frame theory such as frames, Riesz bases, orthonormal bases in Hilbert spaces are other concepts that are introduced in Section 1.2 of this chapter.

1.1 Sequence spaces

Definition 1.1.1. Let X be a normed sequence space, Y be the same as X with a different norm and $A = (a_{n,k})_{n,k\geq 0}$ be an infinite matrix of real or complex numbers. Then, it is said that A defines a matrix mapping from X into Y, and is writing as $A : X \to Y$, if for every sequence $x = (x_k)$ in X the sequence $Ax = \{(Ax)_n\}$, the A-transform of x is in Y where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k, \quad n = 0, 1, \dots .$$

Definition 1.1.2. A non-negative square matrix A is called row stochastic if the sums of all its rows are 1. A column stochastic matrix is the transpose of a row stochastic matrix.

Definition 1.1.3. Let T be a linear operator from a normed space $(X, \|.\|_X)$ into the normed space $(Y, \|.\|_Y)$. If there exists a positive real number κ, such that

$$\|Tx\|_Y \leq \kappa \|x\|_X \quad \forall x \in X, \tag{1.1}$$

then the operator T is called bounded. Norm of T is the smallest number κ that satisfies the inequality (1.1) and is denoted by $\|T\|_{X,Y}$. Moreover, if there exists a positive number l, such that

$$\|Tx\|_Y \geq l \|x\|_X \quad \forall x \in X, \tag{1.2}$$

then the operator T is called bounded away from zero. The lower bound of T is the largest number l satisfies the inequality (1.2) and is denoted by $L_{X,Y}(T)$.

The ordinary Minkowski's inequality with two summand can be generalized to N summand or even a cuntinuum of summand (an integral).

***Proposition* 1.1.4.** *(Minkowski's integral inequality)[38] Fix $p \geq 1$ and let $(X, \mathcal{T}, \mu)$ and (Y, Ω, λ) be σ-finite measure spaces. Let F be $(\mathcal{T} \times \Omega)-$measurable function on $X \times Y$ such that for a.e. $y \in Y$, the function $F^y(x) := F(x, y)$ is in $L^p(\mu)$ and that the function $G(y) := ||F^y||_{L^p(\mu)}$ is in $L^1(\lambda)$. Then the function $F_x(y) := F(x, y) \in L^1(\lambda)$ for a.e. $x \in X$ and the function $H(x) := \int_Y F(x, y) d\lambda(y)$ is in $L^p(\mu)$. Moreover,*

$$\left\| \int\limits_Y F(., y)\, d\lambda \right\|_{L^p(\mu)} \leq \int\limits_Y \|F(., y)\|_{L^p(\mu)} d\lambda.$$

We have the reverse inequality for $0 < p < 1$.

The following theorem is concerning to the integration of functions of two variables. We use this theorem in Chapter 3.

***Theorem* 1.1.5.** *(Fubini's Theorem)[39] Let $(X, \mathcal{T}, \mu)$ and (Y, Ω, λ) be σ-finite measure spaces, and let f be a $(\mathcal{T} \times \Omega)-$measurable function on $X \times Y$.*

1. *If* $0 \leq f \leq \infty$, *and if*

$$\varphi(x) = \int_Y f_x d\lambda, \quad \psi(y) = \int_X f^y d\mu, \qquad (x \in X,\ y \in Y), \tag{1.3}$$

then φ *is* $\mathcal{T}-$*measurable,* ψ *is* $\Omega-$*measurable, and*

$$\int_X \varphi d\mu = \int_{X\times Y} f d(\mu \times \lambda) = \int_Y \psi d\lambda. \tag{1.4}$$

2. *If* f *is complex, and*

$$\varphi^*(x) = \int_Y |f|_x d\lambda \quad and \quad \int_X \varphi^* d\mu < \infty,$$

then $f \in L^1(\mu \times \lambda)$.

3. *If* $f \in L^1(\mu \times \lambda)$, *then* $f_x \in L^1(\lambda)$ *for almost all* $x \in X$, *and* $f^y \in L^1(\mu)$ *for almost all* $y \in Y$; *the functions* φ *and* ψ, *defined by (1.3) a.e., are in* $L^1(\mu)$ *and* $L^1(\lambda)$, *respectively, and (1.4) holds.*

A quasi-norm is similar to a norm that satisfies the norm axioms, except that the triangle inequality is replaced by

$$\|x + y\| \leq K\left(\|x\| + \|y\|\right)$$

for some $K > 1$. We are now ready to introduce the Lorentz space which arises naturally in interpolation theory and finds applications in harmonic analysis, probability theory and functional analysis.

Definition 1.1.6. [22] Let $(X, \mathcal{M}, \mu)$ be a measure space. The Lorentz space $L^{p,q}(X, \mathcal{M}, \mu)$ is the space of all complex-valued measurable functions f on X such that the following quasi-norm is finite

$$|||f|||_{L^{p,q}(X,\mathcal{M},\mu)} = \begin{cases} p^{1/q}\left(\int_0^\infty t^q \mu\{x : |f(x)| > t\}^{q/p} \frac{dt}{t}\right)^{1/q} & 0 < p, q < \infty, \\ \sup\limits_{t>0} t\left(\mu\{x : |f(x)| > t\}\right)^{1/p} & 0 < p < \infty, q = \infty. \end{cases}$$

It is conventional to set $L^{\infty,\infty}(X,\mathcal{M},\mu) = L^{\infty}(X,\mathcal{M},\mu)$, the space of all essentially bounded complex valued measurable functions on X.

The following interesting and valuable proposition states some inclusion relation between the Lorentz spaces.

***Proposition* 1.1.7.** [24] *If* $0 < q_1 \leq q_2 \leq \infty$, *then* $L^{p,q_1}(X,\mathcal{M},\mu) \subseteq L^{p,q_2}(X,\mathcal{M},\mu)$. *In particular,* $L^{p,q_1}(X,\mathcal{M},\mu) \subseteq L^{p,p}(X,\mathcal{M},\mu) \subseteq L^{p,q_2}(X,\mathcal{M},\mu) \subseteq L^{p,\infty}(X,\mathcal{M},\mu)$ *for* $0 < q_1 \leq p \leq q_2 \leq \infty$. *Moreover,* $L^{p,p}(X,\mathcal{M},\mu) = L^{p}(X,\mathcal{M},\mu)$, *where* $L^{p}(X,\mathcal{M},\mu)$ *denotes the space of all complex valued* $\mu-$*measurable functions on* X *whose modulus to the* p*th power is integrable.*

Definition 1.1.8. The Weak L^p space is the Lorentz space $L^{p,\infty}(X,\mathcal{M},\mu)$.

Proposition 1.1.7 showes that the space $Weak\ L^p$ contains the usual space L^p. Also, by the above convention $Weak\ L^{\infty} = L^{\infty}$. The space $Weak\ L^p$ arises naturally in interpolation theory and finds applications in harmonic analysis, probability theory and functional analysis. The following proposition shows that the spaces $Weak\ L^p$ is larger than the usual L^p spaces.

***Proposition* 1.1.9.** [22] *Let* $(X,\mathcal{M},\mu)$ *be a finite measure space and* $0 < p < q$. *Then*

$$L^{q}(X,\mathcal{M},\mu) \subseteq Weak\,L^{q} \subseteq L^{p}(X,\mathcal{M},\mu).$$

***Proposition* 1.1.10.** [22] *Let* $(X,\mathcal{M},\mu)$ *be a measure space. If* $1 < p < \infty$, *then the space Weak* L^p *is normable with the norm*

$$||f||_{Weak\,L^p} = \sup_{E} \frac{1}{(\mu(E))^{1-\frac{1}{p}}} \int_{E} |f|\, d\mu, \tag{1.5}$$

where the sup is taken over all sets $E \in \mathcal{M}$ *with* $0 < \mu(E) < \infty$. *Obviously the space Weak* L^p *is metrizable when* $0 < p < \infty$.

***Proposition* 1.1.11.** [22] *The norm defined with (1.5) is equivalent to the quasi-norm* $|||f|||_{L^{p,\infty}(X,\mathcal{M},\mu)}$ *by the following equivalence relation*

$$|||f|||_{L^{p,\infty}(X,\mathcal{M},\mu)} \leq \ ||f||_{Weak\, L^p} \ \leq \frac{p}{p-1}|||f|||_{L^{p,\infty}(X,\mathcal{M},\mu)}.$$

Consider the set of natural numbers plus zero (the set of all non-negative integers) equipped with counting measure. Then the sequential *weak* ℓ_p space consisting of all complex valued sequences $x = \{x_n\}$ for which

$$|||x|||_{L^{p,\infty}(\mathbb{N}\cup\{0\},2^{\mathbb{N}\cup\{0\}},\#)} = \sup_{t>0} t\left(\#\left\{n \in \mathbb{N}\cup\{0\} : \ |x_n| > t\right\}\right)^{\frac{1}{p}} < \infty,$$

with the norm

$$||x||_{weak\,\ell_p} = \sup_{\substack{B\in\Sigma \\ 0<\#B<\infty}} \frac{1}{(\#B)^{1-\frac{1}{p}}} \sum_{n\in B} |x_n|.$$

Here $\#B$ just means the number of elements in the set B and ℓ_p denote the space of all compelx sequences which are $p-$absolutely summable.

In the rest of this section we introduce some sequence spaces.

Definition 1.1.12. [2] Let $p \in \mathbb{R}\backslash\{0\}$. The space of sequences of $p-$bounded variation is defined as

$$bv_p = \left\{ x = (x_n) \in \mathbb{C} : \ \ \|x\|_{bv_p} := \left(\sum_{n=0}^{\infty} |x_n - x_{n-1}|^p\right)^{\frac{1}{p}} < \infty \right\}.$$

where $x_{-1} = 0$.

It can be shown that bv_p is the space of all real or complex sequences whose $\boldsymbol{\Delta}-$transforms are in the space ℓ_p, where $\boldsymbol{\Delta}$ denotes the matrix $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_{n,k})_{n,k\geq 0}$ defined by

$$\boldsymbol{\Delta}_{n,k} = \begin{cases} (-1)^{n-k} & n-1 \leq k \leq n, \\ 0 & 0 \leq k < n-1,\ k > n. \end{cases} \tag{1.6}$$

Definition 1.1.13. [20] Let $w = (w_n)$ be a sequence of non-negative real numbers and $p \in \mathbb{R}\backslash\{0\}$. The weighted sequence space $\ell_p(w)$ is defined as

$$\ell_p(w) = \left\{ x = (x_n) \in \mathbb{C} : \quad \|x\|_{\ell_p(w)} := \left(\sum_{n=0}^{\infty} w_n |x_n|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

***Lemma* 1.1.14.** *([34], Lemma 1.1) Let $w = (w_n)$ be an increasing non-negative sequence of real numbers and A be a lower triangular matrix with non-negative entries. If*

$$\sup_{n \geq 0} \sum_{k=0}^{n} a_{n,k} = R, \qquad \textit{and} \qquad \inf_{k \geq 0} \sum_{n=k}^{\infty} a_{n,k} = C > 0,$$

then

$$L_{\ell_p(w), \ell_p(w)}(A) \geq R^{\frac{p-1}{p}} C^{1/p}. \qquad (0 < p < 1)$$

Definition 1.1.15. [20] Let $w = (w_n)$ be a sequence of non-negative real numbers, $p \in \mathbb{R}\backslash\{0\}$ and F be partition of non-negative integers. If $F = (F_n)$, where each F_n is a finite interval of non-negative integers and

$$\max\ F_n < \min\ F_{n+1} \qquad (n = 0, 1, 2, ...),$$

then the block weighted sequence space is defined as

$$\ell_p(w, F) = \left\{ x = (x_n) \in \mathbb{C} : \|x\|_{w,p,F} := \left(\sum_{n=0}^{\infty} w_n |< x, F_n >|^p \right)^{\frac{1}{p}} < \infty \right\},$$

where $\langle x, F_n \rangle = \sum_{k \in F_n} x_k$. For a certain I_n such as $I_n = \{n\}$, $I = (I_n)$, is a partition of non-negative integers, $\ell_p(w, I) = \ell_p(w)$, and also $\|x\|_{w,p,I} = \|x\|_{\ell_p(w)}$.

Definition 1.1.16. [32] Let $w = (w_n)$ be a sequence of non-negative real numbers, $p \in \mathbb{R}\backslash\{0\}$ and $0 < \theta < 1$. The Euler weighted sequence space of order θ is defined by

$$e^{\theta}_{w,p} = \left\{ (x_n) \in \mathbb{C} : \|x\|_{e^{\theta}_{w,p}} := \left[\sum_{n=0}^{\infty} w_n \left| \sum_{k=0}^{n} \binom{n}{k} (1-\theta)^{n-k} \theta^k x_k \right|^p \right]^{\frac{1}{p}} < \infty \right\}.$$

It can be shown that $e^{\theta}_{w,p}$ is the spaces of all sequences such that $E(\theta)$-transforms of them are in the space $\ell_p(w)$, where $E(\theta)$ is the operator involved in the method of Euler means of order θ defined by the matrix $E(\theta) = (e_{n,k}(\theta))_{n,k\geq 0}$, where

$$e_{n,k}(\theta) = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} (1-\theta)^{n-k}\theta^k & 0 \leq k \leq n \\ 0 & k > n, \end{cases} \tag{1.7}$$

for all $n, k \in \mathbb{N}$. Obviously, $E(\theta\theta') = E(\theta)E(\theta')$ and $\|x\|_{e^{\theta}_{w,p}} = \|E(\theta)x\|_{\ell_p(w)}$.

The following proposition shows the existence of a linear bijection between the spaces $e^{\theta}_{w,p}$ and $\ell_p(w)$ which is norm preserving.

***Proposition* 1.1.17.** *([32], Theorem 2.2) The sequence space $e^{\theta}_{w,p}$ is linearly isomorphic to the space $\ell_p(w)$, i.e. $e^{\theta}_{w,p} \cong \ell_p(w)$.*

***Proposition* 1.1.18.** *([32], Theorem 2.3) Let the weight sequence $w = (w_n)$ be increasing. Then the inclusion $e^{\theta}_{w,p} \subseteq \ell_p(w)$ holds for $0 < p < 1$.*

***Lemma* 1.1.19.** *Let the weight sequence $w = (w_n)$ be decreasing. Then the inclusion $\ell_p(w) \subseteq e^{\theta}_{w,p}$ strictly holds for $1 \leq p < \infty$.*

Proof. Let $x = \{x_k\} \in \ell_p(w)$ and $y = \{y_k(\theta)\}$ be the $E(\theta)$-transform of the sequence x, i.e.

$$y_k(\theta) = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} (1-\theta)^{k-j}\theta^j x_j.$$