بسم الله الرحمن الرحيم

IN THE NAME OF GOD

INVERSE SEMIGROUP ACTIONS AND C*-CROSSED PRODUCTS BY PARTIAL ACTIONS

BY

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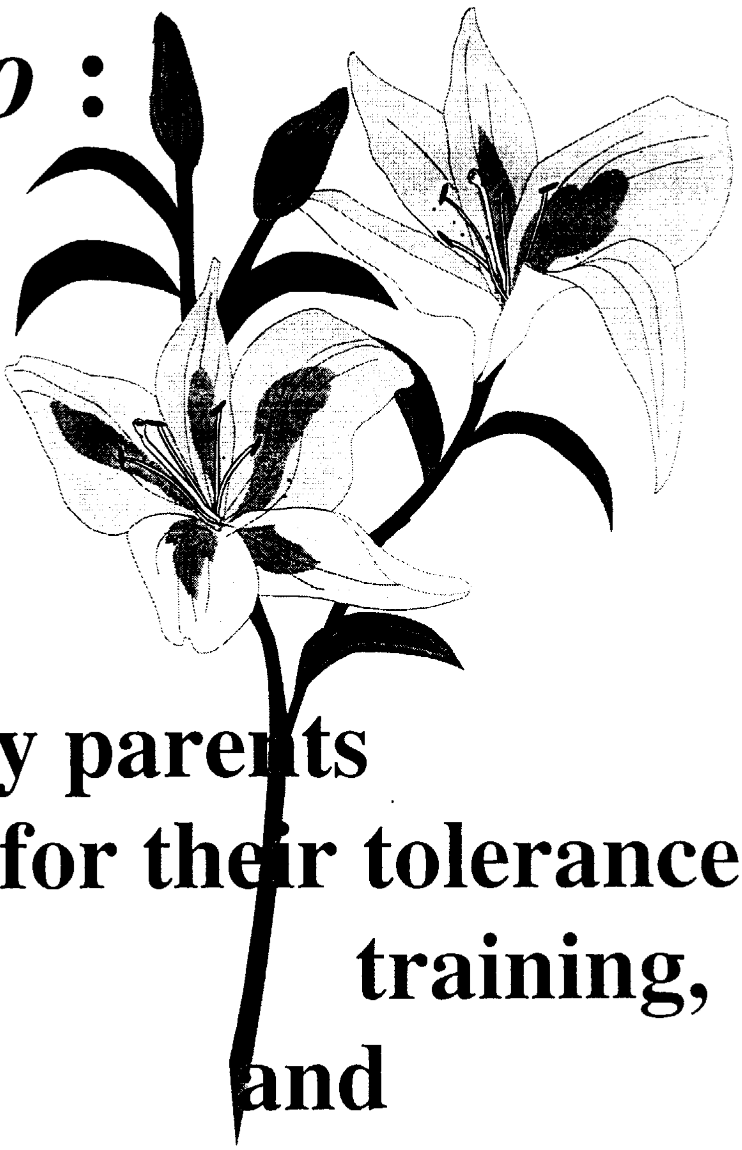

My parents
for their tolerance,
training,
and
patient colaboration.

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No man climbs a mountain alone. To be human is to be dependent.

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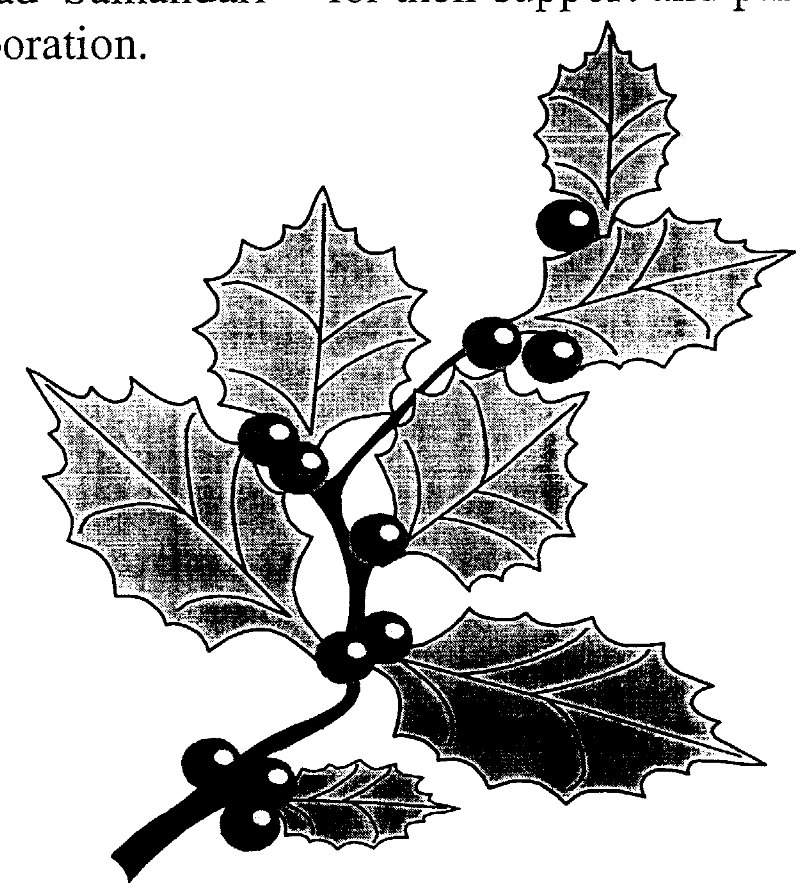

Abstract

Inverse Semigroup Actions and C*-crossed Products by Partial Actions

by

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Recently the notion of a partial crossed product of a C*-algebra by the group of integers was defined by R. Exel in [12]. Roughly, the automorphism used in the definition of a crossed product of a C*-algebra A by the group of integers was replaced by an isomorphism between two closed two-sided ideals of A and was called a partial automorphism. The theory of partial crossed products also can be generalized to discrete groups defined by McClanahan [26].

Our development is based upon another generalization of group actions. In the definition of partial actions we also use the inverse semigroup of partial automorphisms instead of the automorphism group of the C*-algebra.

There is an intimate connection between partial crossed products and crossed products by inverse semigroup actions. Our main goal in this thesis is to explore this connection, showing that, every partial crossed product is isomorphic to a crossed product by an inverse semigroup action. In particular, with a partial action α of a discrete group G on a C*-algebra A, we can define an inverse semigroup S and an action β of S as in Theorem 4.4.2, such that the crossed products $A \times_\alpha G$ and $A \times_\beta S$ are isomorphic.

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Chapter 1

Elementary Properties of C*-algebras

In this chapter we discuss an important class of Banach algebras, termed C^*-algebras. Section 1 is concerned with basic properties of C^*-algebras. we investigate some Examples of C^*-algebras and also some Banach algebras which are not C^*-algebras. States and pure states are study in section 2.

Section 3 is devoted to the notion of approximate identity, an special net of self-adjoint elements of a C^*-algebra. The main goal of section 4 is to introduce the connection between a C^*-algebra A and algebras $C(X)$ or $B(H)$. At the end of this section we define some representations, such as universal, non-degenerate and faithful representations.

Typical references that we use them in this chapter, are [24] and [21].

1.1 C*-algebras

This section deals with some definitions and basic properties of C^*-algebras. A C^*-algebra, is a Banach algebra with an involution that satisfies the C^*-condition. An interesting Example of C^*-algebra is the matrix algebra. At the end of this section we list some important elements of a C^*-algebra.

1.1.1 Preliminaries on C*-algebra

Definition 1.1.1 An *algebra* over $\mathbb{F}$ is a vector space A over $\mathbb{F}$ that also has multiplication defined on it that makes A into a ring such that:

$$\alpha(ab) = (\alpha a)b = a(\alpha b),$$

for all a, b in A and α in $\mathbb{F}$.

A subspace B of A is called *subalgebra* of A if it is an algebra with multiplication defined on A.

Definition 1.1.2 By an *involution* on an algebra A, we mean a mapping $x \longrightarrow x^*$, from A into A such that:

i) $(\alpha x + y)^* = \overline{\alpha} x^* + y^*$;

ii) $(xy)^* = y^* x^*$;

iii) $(x^*)^* = x$,

whenever x, y are in A, α is in $\mathbb{C}$ and $\overline{\alpha}$ denotes the complex conjugation of α.

A subset B of A is said to be *self-adjoint* if it contains the adjoint of each of its members. A self-adjoint subalgebra of A is termed a *$*$-subalgebra*. An algebra with an involution is called *$*$-algebra*.

Definition 1.1.3 If X is a vector space over $\mathbb{F}$, a *norm* on X is a function $P : X \longrightarrow [0, \infty)$ having the properties:

i) $P(x + y) \leq P(x) + P(y)$ for all x, y in X;

ii) $P(\alpha x) = |\alpha| P(x)$ for all α in $\mathbb{F}$ and x in X;

iii) $x = 0$ if $P(x) = 0$.

Usually a norm is denoted by $\|.\|$. Note that if P has properties (i) and (ii), it's called a *seminorm* on X.

Definition 1.1.4 A *normed space* is a pair $(X, \|.\|)$ where X is a vector space and $\|.\|$ is a norm on X. A *Banach space* is a normed space that is complete with respect to the metric defined by the norm.

The following Theorem gives necessary conditions for extension of a function.

Theorem 1.1.5 *(Extension by continuity). Let X be a normed space, Y be a Banach space, X_0 be a dense linear subspace of X and T_0 a bounded linear operator from X_0 to Y. Then T_0 has a unique extension to a bounded linear operator T from X to Y, moreover $\|T\| = \|T_0\|$.*

Proof: See 7L of [34].

Definition 1.1.6 A *Banach algebra* is an algebra A over $\mathbb{F}$ that has a norm $\|.\|$ relative to which A is a Banach space and such that:

$$\|ab\| \leq \|a\|\|b\|,$$

for all a, b in A.

If a Banach algebra A has an identity e, i.e., $ae = ea = a$ for all a in A, then it is assumed that $\|e\| = 1$.

The content of the next Proposition is that if A does not have an identity it is possible to find a Banach algebra A_1, that contains A, that has an identity, and is such that

$$dim(A/A_1) = 1$$

Proposition 1.1.7 *If A is a Banach algebra without an identity, let*

$$A_1 = A \times \mathbb{F}$$

Define algebraic operations on A_1 by:

i) $(a, \alpha) + (b, \beta) = (a + b, \alpha + \beta)$;

ii) $\beta(a, \alpha) = (\beta a, \beta\alpha)$;

iii) $(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$,

and norm by:

$\|(a, \alpha)\| = \|a\| + |\alpha|$.

Then A_1 with this norm is a Banach algebra with identity (0,1) and $a \longmapsto (a, 0)$ is an isometric isomorphism of A into A_1.

Proof: See Proposition 1.1.3 of [33].

Now we are going to define a C*-algebra.

Definition 1.1.8 A *C^*-algebra* is a complex Banach $*$-algebra A that satisfies the condition:

$$\|a^*a\| = \|a\|^2 \quad (a \in A).$$

The above condition is called C^*-condition.

Some times a C*-algebra is denoted by $(A, \|.\|, *)$ whenever $*$ and $\|.\|$ are the involution and norm on A which make it to a C*-algebra.

Proposition 1.1.9 *If A is a C^*-algebra and a is in A, then $\|a^*\| = \|a\|$ and $\|a^*a\| = \|aa^*\|$.*

Proof: Note that $\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$, so $\|a\| \leq \|a^*\|$. Since $a = a^{**}$, substituting a^* for a in this inequality gives $\|a^*\| \leq \|a\|$. Also

$$\|a^*a\| = \|a\|^2 = \|a^*\|^2 = \|a^{**}a^*\| = \|aa^*\|.$$

The following Proposition is needed in the definition of left regular representation.

Proposition 1.1.10 *If A is a C^*-algebra and a is in A, then:*

$$\|a\| = \sup\{\|ax\| : x \in A, \|x\| \leq 1\}.$$

Proof: Let $\alpha = \sup\{\|ax\| : x \in A, \|x\| \leq 1\}$. Then $\|ax\| \leq \|a\|\|x\|$ for any x in A, hence $\alpha \leq \|a\|$. If $x = a^*/\|a\|$, then $\|x\| = 1$. For such x, $\|ax\| = \|a\|$, and so $\alpha = \|a\|$.

Definition 1.1.11 For a in A, define $L_a : A \longrightarrow A$ by $L_a\ (x) = ax$. By preceding Proposition, L_a is in $B(A)$ and $\|L_a\| = \|a\|$. If $\rho : A \longrightarrow B(A)$ is defined by $\rho(a) = L_a$ then ρ is a homomorphism and an isometry. This is. A is isometrically isomorphic to a subalgebra of $B(A)$. The map ρ is called the *left regular representation* of A.

1.1.2 Examples of C*-algebra

Here we will give some Examples of C^*-algebras.

Example 1.1.12 *The simplest Example of a C^*-algebra is $\mathbb{C}$. In this algebra we have $\alpha^* = \overline{\alpha}$ and $\|\alpha\| = |\alpha|$ for all α in $\mathbb{C}$.*

Example 1.1.13 *The set of all n-tuples with complex coordinate,* $\mathbb{C}^n$*, is a* C^**-algebra. Norm, multiplication and involution are defined as follow:*

$$\|(c_1, c_2, ..., c_n)\| = \max\{|c_i| : i = 1, 2, ..., n\},$$

$$(c_1, c_2, ..., c_n)(c_1', c_2', ..., c_n') = (c_1c_1', c_2c_2', ..., c_nc_n'),$$

$$(c_1, c_2, ..., c_n)^* = (\overline{c_1}, \overline{c_2}, ..., \overline{c_n}).$$

It is not difficult to show that $\mathbb{C}^n$ *is a Banach* $*$*-algebra with respect to the above norm. We only prove the* C^**-condition:*

$$\begin{aligned}
\|(c_1, c_2, ..., c_n)(c_1, c_2, ..., c_n)^*\| &= \|(c_1\overline{c_1}, c_2\overline{c_2}, ..., c_n\overline{c_n})\| \\
&= \max\{|c_i\overline{c_i}| : i = 1, 2, ..., n\} \\
&= \max\{|c_i|^2 : i = 1, 2, ..., n\} \\
&= (\max\{|c_i| : i = 1, 2, ..., n\})^2 \\
&= \|(c_1, c_2, ..., c_n)\|^2.
\end{aligned}$$

The set of matrix algebras play an important role in the theory of finite dimensional C^*-algebras. For more on this connection see Theorems (2.2.13) and (2.2.14).

Example 1.1.14 (*Matrix algebra*). *Given an integer* $n \geq 1$ *and the Hilbert space* $\mathbb{C}^n$*, we identify the algebra* $B(\mathbb{C}^n)$ *with the algebra* $M_n(\mathbb{C})$ *of n-by-n complex matrices. Thus* $M_n(\mathbb{C})$ *is a* $C^* -$ *algebra of operator on* $\mathbb{C}^n$*, the involution is given by:*

$$(a)^*_{i,j} = \overline{a_{j,i}}$$

for all a *in* $M_n(\mathbb{C})$ *and* i, j *in* $\{1, 2, ..., n\}$*, and the norm is given by:*

$$\|a\| = \sup\{\|a(x)\| : \quad x \in \mathbb{C}^n, \quad \|x\| \leq 1\} = \sqrt{\max_{1\leq j\leq n} \mu_j}$$

where $\mu_1, \mu_2, \ldots, \mu_n$ *denote the eigenvalues of* a^*a.

For see $\|a^*a\| = \|a\|^2$, *we note that, if* $(A)_{n\times n}$ *is a matrix,* λ^2 *is an eigenvalue of* A^2 *if* λ *is an eigenvalue of* A. *Then*

$$\|a^*a\| = \sqrt{\max_{1\leq j\leq n} \lambda_j} = \sqrt{\max_{1\leq j\leq n} \mu_j^2} = \max_{1\leq j\leq n} \mu_j = \|a\|^2,$$

where λ_i *and* μ_j *are eigenvalues of* $a^*aa^*a = (a^*a)^2$ *and* a^*a, *respectively, for* $1 < i < n$.

Perhaps one think, there is another norm on $M_n(\mathbb{C})$ that makes it to a C*-algebra. but the next Lemma shows that it is not true.

Lemma 1.1.15 *On the involutive algebra* $M_n(\mathbb{C})$, *the only norm,* $\|.\|$, *such that* $\|aa^*\| = \|a\|^2$ *for all* a *in* $M_n(\mathbb{C})$, *is the above norm.*

Proof: See Lemma 2.6 of [21].

It is not deficult to see that, if $(A, \|.\|_1, *), (A, \|.\|_2, *)$ are C*-algebras then $\|.\|_1 = \|.\|_2$.

Now let us show that every multiplier algebra is a C*-algebra. For this we must define first, a double centralizer for a C*-algebra.

Definition 1.1.16 A *double centralizer* for a C^*-algebra A is a pair (L, R) of bounded linear maps on A. such that for all a, b in A

$$L(ab) = L(a)b, \quad R(ab) = aR(b) \quad and \quad R(a)b = aL(b).$$

Example 1.1.17 *(multiplier algebra) If* A *is a* C^**-algebra. the set* $M(A)$ *of all double centralizers of* A *is a* C^**-algebra with the following structures:*

$$\alpha(L_1, R_1) + (L_2, R_2) = (\alpha L_1 + L_2, R_1 + R_2),$$