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REFLEXIVITY AND SUPERCYCLICITY OF SOME CLASSES OF LINEAR OPERATORS

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IN THE NAME OF GOD

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To Karim and Saeed

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ABSTRACT

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In the present thesis, by $\mathcal{H}$ we mean a separable infinite dimensional complex Hilbert space, and $B(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$.

The thesis is organized as follows:

In the first part, we consider the class of m-isometric operators, which are, in some sense, a generalization of isometries. After observing some easy properties of m-isometric operators, we shall characterize all 3-isometric unilateral weighted shift operators, which are not 2-isometric, in terms of their weight sequences. One of the most interesting results is the identification of the behavior of orbits of m-isometries. We will prove that the orbit of every vector under an m-isometric operator is eventually increasing. This leads us to some nice consequences,some of which are as follows:

(1) power bounded m-isometries are isometries;

(2) an m-isometric operator is never supercyclic.

We also consider the subject of weak hypercyclicity of these operators, and prove that no m-isometry is weakly hypercyclic.

The second part of the thesis is devoted to reflexivity of operators. We are going to prove the reflexivity of a hyponormal operator T whose spectrum is $\{z : |z| \leq r(T)\}$, where $r(T)$ denotes the spectral radius of T. Another result is to prove the reflexivity of contractions whose spectrum fills the closed unit disc. We also derive an easy proof for a result of Foias and Pearcy which states that every unilateral or bilateral weighted shift T with $\|T\| = r(T)$ is reflexive. Afterwards, we will show that all non-negative powers of an injective unilateral weighted shift operator such that the point spectrum of whose adjoint has a nonzero element, are reflexive. Furthermore, all integer powers of an invertible bilateral weighted shift operator are reflexive. At last, we will prove that every positive integer power of a unilateral weighted shift that is an m-isometry, is reflexive.

Contents

Chapter 1

INTRODUCTION

1 INTRODUCTION

One of the most intractable unsolved problems in the theory of Hilbert space operators is the invariant subspace problem. This problem deals with the question of whether every operator on a separable infinite dimensional Hilbert space has a nontrivial (closed) invariant subspace; that is, an invariant subspace which is neither (0) nor the whole space. The Banach space analogue has been answered negatively by P. Enflo [27], C. Read [57 ,58], and B. Beauzamy [8]. Some authors are interested in closed invariant subsets other than $\{0\}$ and the whole space which is a stronger condition on operators.

However, the Hilbert space case is still unsolved, although many deep and innovative techniques have been developed to handle special classes of operators. See, for instance, [56].

Some branches in operator theory has been motivated from this problem. One is various kinds of cyclicity, like hypercyclicity and supercyclicity. In Chapter 2, we will be concerned with discussing these concepts for a special class of operators, called m-isometric operators. Another is reflexivity of operators which is the discussion of Chapter 3. Some parts in the scope of our investigation are appeared in [29, 30, 31]. As we will see, if each nonzero vector in a Hilbert space $\mathcal{H}$ is hypercyclic for an operator T, then T has no nontrivial closed invariant subset. In contrast, reflexive operators have numerous invariant subspaces.

In the present chapter, we shall state some basic definitions and preliminary results which will be used in the other parts of this thesis. Throughout this research, $\mathcal{H}$ denotes a separable infinite dimensional complex Hilbert space, unless specifically stated, and $B(\mathcal{H})$ referres to the algebra of all bounded linear operators on $\mathcal{H}$. By an operator, we mean a bounded linear operator. Also, some familiarity with elementary facts about operator theory will be assumed.

1.1 Some Basic Facts about Operators

Let T be an operator on a Hilbert space $\mathcal{H}$. The *spectrum* of T, denoted by $\sigma(T)$, consists of all complex numbers λ such that $T - \lambda I$ is not boundedly invertible. If T^* is the *adjoint* of an operator T, it is well-known that

$$\sigma(T^*) = \sigma(T)^* := \{\lambda \in \mathbb{C} : \overline{\lambda} \in \sigma(T)\}.$$

The *spectral radius* of T is defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$, which satisfies

$$r(T) = \lim_{n \longrightarrow \infty} \|T^n\|^{\frac{1}{n}} \leq \|T\|.$$

The *point spectrum* and the *approximate point spectrum* of an operator T are parts of the spectrum. They are denoted by $\sigma_p(T)$ and $\sigma_{ap}(T)$, respectively. The point spectrum of T is, by definition, the set of all scalars λ such that $ker\ (T - \lambda) \neq (0)$. Furthermore, $\sigma_{ap}(T)$ consists of all $\lambda \in \mathbb{C}$ for which there is a sequence $\{h_n\}_n$ in $\mathcal{H}$ such that $\|h_n\| = 1$ for all n and $\|(T - \lambda)h_n\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Recall that an operator T is *bounded below*, if there is a constant $c > 0$ such that $\|Th\| \geq c\|h\|$, for all h in $\mathcal{H}$.

Proposition 1.1.1 *If T is an operator on $\mathcal{H}$ and $\lambda \in \sigma(T)$, then the following assertions are equivalent.*

(a) *λ is not in $\sigma_{ap}(T)$.*

(b) *The operator $T - \lambda$ is bounded below.*

(c) *ran $(T - \lambda)$ is closed and ker $(T - \lambda) = (0)$.*

Proof. See [19, Proposition 6.4 (VII)]. □

Example 1.1.2 If T is a compact operator on an infinite dimensional Hilbert space $\mathcal{H}$, then $0 \in \sigma(T)$ and either $\sigma(T)$ is finite or consists of a sequence converging to zero.

Note that if T is a compact operator then $\sigma(T)\backslash\{0\} \subseteq \sigma_p(T)$ [19, Theorem 7.1 (VII)].

To continue, we recall that if X_1 and X_2 are Banach spaces, then $\mathrm{X}_1 \oplus_1 \mathrm{X}_2$ denotes the Banach space of all $x = h_1 \oplus h_2$ with norm defined by $\|x\| = \|h_1\| + \|h_2\|$.

The next result is an application of the closed graph theorem which is sometimes used to prove certain linear transformations are bounded. Recall that by the closed graph theorem, if the graph of a linear transformation is closed, then it is continuous.

Proposition 1.1.3 *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a linear transformation, then the graph of T, i.e., $\{h \oplus Th \in \mathcal{H}_1 \oplus_1 \mathcal{H}_2 : h \in \mathcal{H}_1\}$ is closed if and only if whenever $h_n \longrightarrow 0$ and $Th_n \longrightarrow k$, it must be that $k = 0$.*

Proof. It is straightforward. □

We now turn to another result which is useful in the subsequent chapters.

Theorem 1.1.4 *Suppose that $T \in B(\mathcal{H})$. Then ran T is closed if and only if ran T^* is closed.*

Proof. See [19, Theorem 1.10 (VI)]. □

In many situations, we deal with special subspaces of a Hilbert space, which are introduced in the next definition.

Definition 1.1.5 Let T be an operator in $B(\mathcal{H})$. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is called an *invariant subspace* of T, if $T\mathcal{M} \subseteq \mathcal{M}$. Also, $\mathcal{M}$ is called a *reducing subspace*, if it is invariant under both T and T^*.

This section concludes with a glance at the Riesz functional calculus for an operator in $B(\mathcal{H})$. For further discussion on this subject, we refer to [19].

Definition 1.1.6 Let T be in $B(\mathcal{H})$. By $Hol(T)$ we mean the set of all complex valued functions that are analytic in a neighborhood of $\sigma(T)$.

Theorem 1.1.7 *Suppose that $T \in B(\mathcal{H})$. Then there exists a unique algebra homomorphism $\tau : Hol(T) \longrightarrow B(\mathcal{H})$ with the following properties.*

(a) *If $f(z) \equiv 1$, then $\tau(f) = I$.*

(b) *If $f(z) = z$ for all z, then $\tau(f) = T$.*

(c) *If $f,\ f_1,\ f_2,\ldots$ are analytic functions on an open set G such that $\sigma(T) \subseteq G$ and $f_n(z) \longrightarrow f(z)$ uniformly on compact subsets of G, then $\tau(f_n) \longrightarrow \tau(f)$.*

Proof. See [19, Proposition 4.8 (VII)]. □

Notation. The operator $\tau(f)$, introduced in the above theorem is represented by $f(T)$.

1.2 Strong and Weak Operator Topologies on $B(\mathcal{H})$

In this section, we will make a brief study of various kinds of topologies on $B(\mathcal{H})$. Note that since $B(\mathcal{H})$ is a normed space, it admits the norm topology.

But there are other topologies on $B(\mathcal{H})$ which results in serious studies of this space. We, especially, make a use of them in the last chapter.

Definition 1.2.1 The *strong operator topology* (SOT) on $B(\mathcal{H})$ is the topology defined by the collection of seminorms $\{p_h : h \in \mathcal{H}\}$ where $p_h : B(\mathcal{H}) \longrightarrow \mathbb{C}$ is given by $p_h(T) = \|Th\|$.

According to this definition, for an operator $T_0 \in B(\mathcal{H})$, the family of sets of the form

$$U(T_0 : h; \varepsilon) = \{T \in B(\mathcal{H}) : \|(T - T_0)h\| < \varepsilon\},$$

where $\varepsilon > 0$ and h is in $\mathcal{H}$, constitutes a subbase of open neighborhoods of T_0 in SOT.

Definition 1.2.2 The *weak operator topology* (WOT) on $B(\mathcal{H})$ is the topology defined by the collection of seminorms $\{p_{h,k} : h, k \in \mathcal{H}\}$ where $p_{h,k} : B(\mathcal{H}) \longrightarrow \mathbb{C}$ is given by $p_{h,k}(T) = |\langle Th, k\rangle|$.

In this topology, an operator T_0 in $B(\mathcal{H})$ has a subbase of neighborhoods consisting of all sets of the type

$$U(T_0 : h, k; \varepsilon) = \{T \in B(\mathcal{H}) : |\langle (T - T_0)h, k\rangle| < \varepsilon\},$$

where $\varepsilon > 0$ and h, k are in $\mathcal{H}$.

The following proposition illustrates the convergence in SOT and WOT on $B(\mathcal{H})$.

Proposition 1.2.3 *Let $\{T_i\}_i$ be a net in $B(\mathcal{H})$.*

(a) *$T_i \longrightarrow T$ (SOT) if and only if $\|T_i h - Th\| \longrightarrow 0$ for every h in $\mathcal{H}$.*

(b) *$T_i \longrightarrow T$ (WOT) if and only if $\langle T_i h, k\rangle \longrightarrow \langle Th, k\rangle$ for every $h,\ k$ in $\mathcal{H}$.*

Proof. See [19]. □

The next theorem is crucial when studying convex subsets of $B(\mathcal{H})$.

Theorem 1.2.4 *The weak operator closure of a convex subset of $B(\mathcal{H})$ coincides with its strong operator closure.*

Proof. See [45, Theorem 5.1.2]. □

1.3 Weighted Shift Operators

Weighted shift operators provide a good source of plenty of examples in many branches of operator theory. To get an explicit description of them, we begin with the following definitions.

Definition 1.3.1 A (not necessarily bounded) operator T on $\mathcal{H}$ is called a *unilateral weighted shift*, if there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ and a sequence of scalars $\{w_n\}_{n=0}^{\infty}$ such that $Te_n = w_n e_{n+1}$, for all $n \geq 0$.

Definition 1.3.2 A (not necessarily bounded) operator T on $\mathcal{H}$ is called a *bilateral weighted shift*, if there is an orthonormal basis $\{e_n : n \in \mathbb{Z}\}$ and a bilateral sequence $\{w_n : n \in \mathbb{Z}\}$ such that $Te_n = w_n e_{n+1}$, for all $n \in \mathbb{Z}$.

The sequence $\{w_n\}_n$ is called the weight sequence of T. Note that a weighted shift operator is bounded if and only if its weight sequence forms a bounded sequence.

We remark that it will always suffice to assume that the weight sequence consists of nonnegative scalars, due to the following result.

Proposition 1.3.3 *If T is a unilateral (resp. bilateral) weighted shift operator with weight sequence $\{w_n\}_n$, then T is unitarily equivalent to a unilateral (resp. bilateral) weighted shift operator with a nonnegative weight sequence.*

Proof. See [21, Page 53]. □

Theorem 1.3.4 *If T is a bounded weighted shift operator then*

$$\|T^n\| = \sup_k |w_k w_{k+1} \dots w_{k+n-1}|, \qquad n = 1, 2, \cdots.$$

Proof. The result clearly follows from the fact that

$$T^n e_k = (w_k w_{k+1} \dots w_{k+n-1}) e_{k+n},$$

for every k. □

Theorem 1.3.5 *If T is a bounded bilateral weighted shift operator then*

$$T^* e_n = \overline{w_{n-1}} e_{n-1},$$

for all n. Also, if $T \in B(\mathcal{H})$ is a unilateral weighted shift operator, then

$$T^* e_n = \begin{cases} \overline{w_{n-1}} e_{n-1} & (n \geq 1), \\ 0 & (n = 0). \end{cases}$$

Proof. See [67, Page 52]. □

Recall that an operator T is said to be *power bounded*, if $\sup\{\|T^n\| : n = 0, 1, \dots\} < \infty$.

Theorem 1.3.6 *Every power bounded weighted shift operator is similar to a contraction.*

Proof. See [67, Page 55]. □

Recall that an operator T is called *hyponormal*, if $T^*T - TT^*$ is a positive operator; or equivalently, if $\|Th\| \geq \|T^*h\|$, for every vector h in $\mathcal{H}$. For a hyponormal operator T, we always have $\|T\| = r(T)$ (see [21]).

Theorem 1.3.7 *A weighted shift operator is hyponormal if and only if its weight sequence is increasing.*

Proof. See [21, Proposition 6.6] or [67, Page 83]. □

Shield's paper [67] is a good reference on the spectrum of weighted shift operators.

Theorem 1.3.8 *If T is a unilateral weighted shift operator, then*

$$\sigma(T) = \{z : |z| \leq r(T)\}.$$

Theorem 1.3.9 *Suppose that T is a bilateral weighted shift operator. Then the following assertions are true.*

(a) *If T is invertible, then*

$$\sigma(T) = \{z : r(T^{-1})^{-1} \leq |z| \leq r(T)\}.$$

(b) *If T is not invertible, then*

$$\sigma(T) = \{z : |z| \leq r(T)\}.$$

1.4 Another View of Weighted Shifts

In this section, we describe another way of introducing injective weighted shift operators.

Recall that a (not necessarily bounded) operator T on $\mathcal{H}$ is said to shift (forward) an orthogonal basis $\{f_n\}_n$, if $Tf_n = f_{n+1}$, for every n. Let $\{e_n\}_n$ be an orthonormal basis for $\mathcal{H}$ and T be a weighted shift operator with positive weight sequence $\{w_n\}_n$. Then T shifts the orthogonal basis $\{f_n\}_n$ defined by