*Ferdowsi University of Mashhad*

# Fragmentability and Approximation

by

*Ali-reza Kamel Mirmostafaee*

*Supervised by*

*A. Niknam*

*1997*

# Preface

In this thesis, some geometrical aspect of Banach spaces, i.e, fragmentability and approximation are discussed. The notion of fragmentability of a topological space is appeared in a paper of Jayne and Rogers [25]. It turned out that this notion have interesting connection with other properties of Banach spaces, and attracted the attention of recent research workers ( see e.g [20]-[24], [37], [31], [32], [45]-[47] ). In chapter 1, charactrizations of fragmentability, which are obtained by Namioka [37], Ribarska [45] and kenderov-Moors [32], are given. Also the connection between fragmentability and its variants and other topics in Banach spaces such as analytic spaces, the Radon-Nikodym property, differentiability of convex functions, Kadec renorming are discussed.

In chapter 2, we use game charactrization of frgmentability of Kenderov-Moors to construct a large class of non-fragmentable Banach spaces. In particular, we will show that if a compact Hausdorff space $X$ contains a non-trivial converging sequence, then $(C(\tilde{X})/C(X), weak)$ is not fragmented by any metric, where $\tilde{X}$ is the Gleason extremally disconnected space corresponding to $X$.

Pedersen proved that if $A$ is a $C^*$-algebra with unit, $B$ a Chebyshev $C^*$-subalgebra of $A$ then either $B = A$, $B = \mathbb{C}1$ or else $A = M_2(\mathbb{C})$, $2 \times 2$ matrices, and $B$ is isomorphic to the algebra of diagonal matrices. We extend his result for $JB$-algebras, in chapter 3, we will show that if $B$ is a Chebyshev subalgebra of a unital $JB$-algebra $A$, then either $B$ is a trivial subalgebra of $A$ or $A$ has a representation of the form $H \oplus \Re 1$, where $H$ is a Hilbert space. This connects algebraic and geometric properties of a $JB$-algebra.

A famous problem in approximation theory is whether or not sets having unique farthest point property are singletons. Klee [33] proved that any positive answer to this problem in Hilbert spaces implies that all Chebyshev sets are convex. The problem is considered in chapter 4 for two special cases, i.e, it is shown that every uniquely remotal subset of an alternative $JB$-algebra or $\ell^\infty$-sum of Banach spaces is a singleton.

I take this opportunity to thank my supervisor Professor A. Niknam sincerely for his great patience, unfailing encouragment and helpful commands throughout the period of my education.

Thanks are also due to Professor M.A.Pourabdollah for all that I have learend from him during my education and his helpful suggestions.

I am also debtor to Professor P.S. Kenderov for his hospitality and furfilled discussion on fragmentability, when I was a guest in Bulgarian Academy of Sciences.

Alireza - Kamel Mirmostafaee

# Contents

# Chapter 1

# Fragmentability and $\sigma$-fragmentability

## 1.1 Introduction

Let $(X, \tau)$ be a topological space and let $\rho$ be a pseudo-metric on $X$. Given $\epsilon > 0$, a non empty subset $A$ of $X$ is said to be fragmented by $\rho$ down to $\epsilon$ if each non-empty subset $B$ of $A$ has a relatively $\tau$-open subset of diameter less than $\epsilon$. The set $A$ is said to be fragmented by $\rho$, if $A$ is fragmented by $\rho$ down to $\epsilon$ for each $\epsilon > 0$. The set $A$ is said to be $\sigma$-fragmented by $\rho$, if for each $\epsilon > 0$, $A$ can be expressed as $A = \bigcup_{n=1}^{\infty} A_n$ with each $A_n$ fragmented by $\rho$ down to $\epsilon$. The notion of fragmentability and its variants have been found to have interesting connection with other topics such as locally uniformly convex and Kadec renorming of Banach spaces and the differentiability of convex function on Banach spaces.

## 1.2 Fragmentability and continuity

A topological space $X$ is called hereditarily Baire if each closed subset of $X$ is a Baire space with respect to the relative topology.

**Lemma 1.2.1** ([37], Lemma1.1 ) *Let* $(X, \tau)$ *be a hereditarily Baire space, and let* $\rho$ *be a metric on* $X$. *Then the following conditions are equivalent:*

*(i)The space* $(X, \tau)$ *is* $\rho$*-fragmented.*

*(ii)For each non-empty closed subset* $A$ *of* $(X, \tau)$ *the set of points of continuity of the identity map* :$(A, \tau) \to (A, \rho)$ *is a dense* $G_\delta$*-subset of* $(A, \tau)$.

*(iii)For each non-empty closed subset* $A$ *of* $X$, *there is a point of continuity for map* :$(A, \tau) \to (A, \rho)$.

A compact Hausdorff space $X$ is said to be *Eberleian compact (EC)* if $X$ is homeomorphic to a weakly compact subset of a Banach space. Each EC space is fragmented by a metric because of the following result (cf [37] ).

**Theorem 1.2.1** *If $K$ is a weakly compact subset of a Banach space $E$, then the identity map* : $(K, \text{weak}) \to (K, \text{norm})$ *admits a point of continuity. Consequently, each weakly compact subset of $E$ is norm-fragmented.*

Let $X$ and $Y$ be Hausdorff topological spaces and $F : X \to Y$ be a multivalued function. $F$ is said to be upper semi-continuous if for every open subset $U$ of $Y$ the set $\{x \in X : F(x) \subset U\}$ is open in $X$. We say that $F$ is usco correspondence if $F$ is upper semi-continuous and $F(x)$ is a nonempty compact subset of $Y$ for every $x \in X$.

The graphs of all usco correspondences from $X$ into $Y$ can be ordered by inclusion. Minimal elements of this relation are graphs of usco correspondenes from $X$ into $Y$ which will be called *"minimal usco"* [10].

Let $F : X \to Y$ be a minimal usco correspondence between the Hausdorff topological spaces $X$ and $Y$. then the following property holds:

For every open subset $U$ of $X$ and every open subset $V$ of $Y$ such that $F(U) \cap V \neq \emptyset$, there is a nonemty open subset $W$ of $U$ wih $F(W) \subset V$.

Applying the natural construction used by Kenderov [30], and Christensen and Kenderov [10], Ribarska [45] proved that :

**Proposition 1.2.1** *Let $B$ be a Baire space, $X$ be a Hausdorff topological space, fragmented by the metric $d$, and $F : B \to X$ be a minimal usco corresponenece from $B$ to $X$. Then there is a dense $G_\delta$ subset $G$ of $B$ so that, for every $x \in G$, the set $F(x)$ is a singleton. Moreover, $F$ is continuous at the points of $G$ , when $X$ is endowed with the topology generated by $d$.*

The next result states the interrelation between fragmentability and minimal mappings ([32], theorem 5.1 ):

**Theorem 1.2.2** *Let $F : Z \to X$ be a minimal mapping between topological spaces $Z$ and $X$. Suppose that $X$ is fragmented by some metric $d$. Then there exists a first Baire category subset $E$ of $Z$ such that, at every $z \in Z \setminus E$, either*

*$F(z) = \emptyset$ or the mapping $F$ is single-valued and usco with respect to $d$.*

A map $f$ of the product $X \times Y$ of topological space $X$ and $Y$ into a topological space $Z$ is said to be *separately continuous* if , for each $(x_0, y_0)$ in $X \times Y$, the maps $x \mapsto f(x, y_0) : X \to Z$ and $y \mapsto f(x_0, y) : Y \to Z$ are continuous. When $f$ is continuous at $(x_0, y_0)$ relative to the product topology, we shall say that $f$ is *jointly continuous* at $(x_0, y_0)$.

A compact Hausdorff space $K$ is said to be in the class $\mathcal{N}^*$ if for each Baire space $Y$, and each separately continuous map $f : Y \times K \to R$, there exists a dense $G_\delta$ subset $D$ of $Y$ such that $f$ is jointly continuous at each point of $D \times K$. One can easily see that $K$ is in $\mathcal{N}^*$, if and only if , whenever $f$ is a continuous map of the Baire space $Y$ into $(C(K), p)$ , where $p$ denotes the pointwise topology on $C(K)$, there exists a dense $G_\delta$ subset $D$ of $Y$, such that $f : Y \to (C(K),$ norm $)$ is continuous at each point of $D$.

The following result is a trivial corollary of lemma 1.2.1.

**Corollary 1.2.1** *If $K$ is a compact Hausdorff space such that $(C(K), p)$ is fragmented by the norm , then $K$ is in the class $\mathcal{N}^*$.*

## 1.3 Relatively open partitioning and fragmentability

A well ordered family $\mathcal{U} = \{U_\zeta : 0 \leq \zeta < \zeta_0\}$ of subsets of the topological space $X$ is said to be a *relatively open partitioning* of $X$ if

(i)$U_0 = \emptyset$ ;

(ii) $U_\zeta$ is contained in $X \backslash (\cup_{\eta<\zeta} U_\eta)$ and is relatively open in it for every $\zeta, 0 < \zeta < \zeta_0$ ;

(iii)$X = \cup_{\zeta<\zeta_0} U_\zeta$.

A family $\mathcal{U}$ of subsets of the topological space $X$ is said to be a $\sigma-$*relatively open partionning* of $X$, if $\mathcal{U} = \cup_{n=1}^{\infty} \mathcal{U}^n$ and $\mathcal{U}^n$, $n = 1, 2, \ldots$, are relatively open partionnings of $X$. $\mathcal{U}$ is said to separate the points of $X$, if whenever $x$ and $y$ are two different elements of $X$, there exists $n$ such that $x$ and $y$ belong to different elements of the partionning $\mathcal{U}^n$.In this case we say that $X$ admits a seperating $\sigma-$relatively open partionning.

The relation between fragmentability and $\sigma$-relatively open partionning is discribed in the following theorem:

**Theorem 1.3.1** ([46], theorem 1.9 ) *The topological space $X$ admits a separating $\sigma-$relatively open partionning , if and only if there exists a metric which fragments $X$.*

We could consider topological spaces which are fragmented by some non-negative function $\lambda : X \times X \to [0, \infty)$ with $\lambda(x, y) = 0$ if and only if $x = y$. Ribarska noted that if there exists such a fragmenting function on $X$, then there exists a metric which fragments it [45].

The following corollary is obtanied by Ribarska by means of theorem 1.3.1 (cf [45], p.247]):

**Corollary 1.3.1** *Let X be a Hausdorff compact space, which admits separating $\sigma-$relatively open partionning. Then there exists a complete metric $\rho$ on $X$, which fragments it and such that the topology of $\rho$ is stronger than the original topology on $X$.*

Ribarska used the notion of $\sigma$-relatively open partionning to prove the following result, which gives us a large class of fragmentable spaces [45]:

**Theorem 1.3.2** *Let X be a Hausdorff compact space which is fragmented by a metric. Then the space $C(X)^*$, endowed with the weak star topology is a fragmented space as well.*

The class of fragmentable spaces is closed under taking subspaces, countable product and perfect images, more precisly, we have:

**Proposition 1.3.1** ( [45] p.250 ) *Let $\mathcal{M}$ be the class of fragmentable spaces.*

*(a)If $X \in \mathcal{M}$ then $X_1 \in \mathcal{M}$ for every $X_1 \subset X$.*

*(b)Let $X \in \mathcal{M}$ and $g$ be a perfect mapping from $X$ onto the topological space $Y$ (i.e. $g$ is a continuous mapping which maps closed subsets of $X$ into closed subsets of $Y$ and $g^{-1}(y)$ is a compact non-empty subset of $X$ for every $y \in Y$), then $Y \in \mathcal{M}$.)*

*(c)From $X = \bigcup_{i=1}^{\infty} X_i$ where all $X_i$ are closed in $X$ and $X_i \in \mathcal{M}, i = 1, 2 \ldots,$ it follows that $X \in \mathcal{M}$.*

*(d) If the spaces $X_i, i = 1, 2, \ldots$ are in $\mathcal{M}$, then their cartesian product $\Pi_{i=1}^{\infty} X_i$ also belongs to $\mathcal{M}$.*

*(e) Let $X \in \mathcal{M}$ and $h$ be a continuous one-one mapping from $Y$ into $X$. Then $Y$ is in $\mathcal{M}$.*

**Corollary 1.3.2** ( [45], p.251) *If $X$ is a fragmentable compact space and $g : X \to Y$ is a continuous (single-valued) mapping onto the topological space $Y$, then $Y$ is a fragmentable space.*

## 1.4 Game charactrization of fragmentability

In [31] the following topological game was used to characterize fragmentability of the space $X$. Two players $\Sigma$ and $\Omega$ select alternatively subsets of $X$. $\Sigma$ starts the game by selecting an