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IN THE NAME OF GOD

CLASSIFICATION OF ANALYTIC CROSSED PRODUCT

BY

M. T. HEYDARI

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AS: EXCELENT

B. Tabatabaie .... B. TABATABAIE, Ph.D., ASSISTANT PROF. OF MATHEMATICS(CHAIRMAN)

K. Seddighi .... K. SEDDIGHI, Ph.D., PROF. OF MATHEMATICS

B. Yousefi .... B. YOUSEFI, Ph. D., ASSTISTANT PROF. OF MATHEMATICS

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ABSTRACT

CLASSIFICATION OF ANALYTIC CROSSED PRODUCT OF C^*– ALGEBRAS.

BY

M.T.HEYDARI

If X_i is a locally compact Hausdorff space and $\phi_i : X_i \longrightarrow X_i$ is a homeomorphism, $i = 1, 2$, then (X_1, ϕ_1) and (X_2, ϕ_2) are said to be conjugate if there is a homeomorphism $\psi : X_2 \longrightarrow X_1$ such that $\psi o \phi_2 = \phi_1 o \psi$.

Let $C_0(X) \times_\phi Z$ be the C^*-crossed product of (X, ϕ).The semi-crossed product associated to (X, ϕ) is the closed subalgebra of the C^*–algebra $C_0(X) \times_\phi Z$, and denoted by $C_0(X) \times_\phi Z_+$.

This dissertition contains three chapters. In chapter one, we will give some basic definitions, theorems and some concepts which we will need later on in our work.

Chapter two is devoted to the crossed product of C^*-algebras.

The semi-crossed product of C^*-algebras, its structure and its classification are disscused in chapter three.

Finally we prove that:

The semi-crossed products $C_0(X_1) \times_{\phi_1} Z_+$ and $C_0(X_2) \times_{\phi_2} Z_+$ are isomorphic as compelex algebras if and only if the pairs (X_1, ϕ_1) and (X_2, ϕ_2) are conjugate.

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Chapter 1

ELEMENTARY C^*-ALGEBRA THEORY

In this chapter we study a special class of Banach-algebras, termed C^*-algebras, the ones that have an involution with properties parallel to those of the adjoint operation on Hilbert space operators. With X a compact Hausdorff space and H a Hilbert space, $C(X)$ and $B(H)$ are examples of C^*-algebras, and so is each norm-closed subalgebra of $B(H)$ that contains the adjoint of each of its members. Two basic representation theorems assert that, up to isomorphism, these are the only examples; every C^*-algebra can be viewed as normed, closed self-adjoint subalgebra of $B(H)$, for an appropriate choice of H, and every abelian C^*-algebra is isomorphic to one of the form $C(X)$.

In section 1 we shall recall the basic definitions.

Section 2 starts with the concepts of positive elements, state, vector state and pure state. At the end of this section we proved that every abelian C^*-algebra is isomorphic to one of the form $C(X)$.

Section 3 is devoted to the representation of a C^*-algebra on a Hilbert space.

1.1 Basics

Definition 1.1.1 If X is a vector space over $\mathbf{F}(= \mathbf{C}\ \ or \mathbf{R})$, a *seminorm*

is a function $p : X \to [0, \infty)$ having the properties:

(a) $p(x+y) \leq p(x) + p(y)$

(b) $p(\alpha x) = |\alpha| p(x)$. $\qquad \alpha \in \mathbf{F};\ \ x, y \in X$

In follows from (b) that $p(0) = 0$, *A norm is a seminorm* p *such that:*

(c) $x = 0$ *if* $p(x) = 0$.

Usually a norm is denoted by $||.||$.

A *normed space* is a pair $(X, ||.||)$, where X is a vector space and $||.||$ is a norm on X.

Definition 1.1.2 *A topological vector space over a field* $\mathbf{F}$ *(TVS) is a vector-space* X *together with a topology such that with respect to this topology*

(a) the map of $X \times X \longrightarrow X$ *defined by* $(x, y) \mapsto x + y$ *is continuous;*

(b) the map of $\mathbf{F} \times X \longrightarrow X$ *defined by* $(\alpha, x) \mapsto \alpha x$ *is continuous.*

It is easy to see that every normed space is a TVS. Suppose X is a vector space and $\mathcal{P}$ is a family of seminorms on X. Let τ be the topology on X that has a subbase the sets $\{x : p(x - x_0) < \varepsilon\}$, where $p \in \mathcal{P}, x_0 \in X$, and $\varepsilon > 0$. Thus a subset V of X is open if and only if for every x_0 in V there are $p_1, p_2, \cdots, p_n$ in $\mathcal{P}$ and $\varepsilon_1, \cdots, \varepsilon_n > 0$ such that $\bigcap_{j=1}^{n} \{x : p_j(x - x_0) < \varepsilon_j\} \subseteq V$. It is not difficult to show that X with this topology is TVS.

A *locally convex space*(LCS) is a TVS whose topology is defined by a family of seminorms $\mathcal{P}$ such that $\bigcap_{p \in \mathcal{P}} \{x : p(x) = 0\} = (0)$. This condition implies that the topology defined by $\mathcal{P}$ is Hausdorff. In fact, suppose that $x \neq y$. Then there is a p in $\mathcal{P}$ such that $p(x - y) \neq 0$, let $p(x-y) > \epsilon > 0$. If $U = \{z : p(x-z) < \epsilon/2\}$ and $V = \{z : p(y-z) < \epsilon/2\}$, then $U \cap V = \emptyset$ and U and V are neighborhoods of x and y, respectively.

Example 1.1.1 *Let X be a normed space and for each x^* in X^*(= the set of all continuous linear functionals on X), define $p_{x^*}(x) = |x^*(x)|$. Then p_{x^*} is a seminorm and if $\mathcal{P} = \{p_{x^*} : x^* \in X^*\}, \mathcal{P}$ makes X into a LCS.*

The topology defined on X by these seminorms is called the weak topology or(wk)topology on$X$; and is often denoted by $\sigma(X, X^*)$.

Example 1.1.2 *Let X be a normed space and for each x in X define $p_x : X^* \longrightarrow [0, \infty)$ by $p_x(x^*) = |x^*(x)|$. Then p_x is a seminorm on X^*, and $\mathcal{P} = \{p_x : x \in X\}$ makes X^* into a LCS. The topology defined by these seminorms is called the weak – star topology or (wk^*) topology on X^*. It is often denoted by $\sigma(X^*, X)$.*

Definition 1.1.3 *If X is a vector space over $\mathbf{F}(\mathbf{C}$ or$\mathbf{R})$, an inner product on X is a function $< .,. >: X \times X \to \mathbf{F}$ such that:*

(a) $< y, x >= \overline{< x, y >}$. (The bar denotes the complex conjugation.)

(b) $< x + y, z >=< x, z > + < y, z >$ if x, y and $z \in X$,

(c) $< \alpha x, y >= \alpha < x, y >$ if x and $y \in X$ and α is a scalar,

(d) $< x, x >\geq 0$ for all $x \in X$,

(e) $< x, x >= 0$ only if $x = 0$

An inner product space is a pair $(X, < .,. >)$, where X is a vector space and $< .,. >$ is an inner product on X.

If $(X, <.,.>)$ is an inner product space, it is a normed space with, $||x|| = <x, x>^{1/2}$.

Definition 1.1.4 *A Hilbert space is a vector space H over $\mathbf{F} = (\mathbf{C}$ or $\mathbf{R})$ together with an inner product $<.,.>$ such that relative to the metric $d(x,y) = ||x-y||$ induced by the norm, H is a complete metric space.*

Let $B(H)$ be the set of all continuous linear transformations from H into H. If $A \in B(H)$, then $||A|| = sup\{||Ah||, ||h|| = 1\}$.

Definition 1.1.5 *If H is a Hilbert space, the weak operator topology(WOT) on $B(H)$ is the locally convex topology defined by the family of seminorms $\{p_{h,k} : h, k \in H\}$, where $p_{h,k}(A) = | <Ah, k> |$.*

The strong operator topology(SOT) is the topology defined on $B(H)$ by the family of seminorms $\{p_h, h \in H\}$ where $p_h(A) = ||A(h)||$.

Definition 1.1.6 *A Banach space is a normed space that is complete with respect to the mertic defined by the norm, $d(x,y) = ||x-y||$. An algebra over $\mathbf{F} = (\mathbf{C}$ or $\mathbf{R})$ is a vector space $\mathcal{U}$ over $\mathbf{F} = (\mathbf{C}$ or $\mathbf{R})$ that also has a multiplication defined on it that makes $\mathcal{U}$ into a ring such that if $\alpha \in \mathbf{F} = (\mathbf{C}$ or $\mathbf{R})$ and $A, B \in \mathcal{U}, \alpha(AB) = (\alpha A)B = A(\alpha B)$.*

Definition 1.1.7 *A Banach algebra is an algebra $\mathcal{U}$ over $\mathbf{F} = (\mathbf{C}$ or $\mathbf{R}$ that has a norm $||.||$ relative to which $\mathcal{U}$ is a Banach space and such that for all $A, B \in \mathcal{U}, ||AB|| \leq ||A||||B||$.*

If $\mathcal{U}$ is a Banach algebra, an involution is a map $A \mapsto A^*$ of $\mathcal{U}$ into $\mathcal{U}$ such that the following properties hold for A and B in $\mathcal{U}$ and $\alpha \in \mathbf{C}$:

(i) $(A^*)^* = A$

(ii) $(AB)^* = B^*A^*$

(iii) $(\alpha A + B)^* = \overline{\alpha}A^* + B^*$.

Definition 1.1.8 *A $C^* -$ algebra is a Banach algebra $\mathcal{U}$ with an involution such that for every A in $\mathcal{U}$, $||A^*A|| = ||A||^2$.*

All C^*-algebras in this dissertation are assumed to has unitary unless the contrary is specified.

This last condition ensures that the involution in a C^*-algebra preserves norm(and is therefor continuous); for $||A||^2 = ||A^*A|| \leq ||A^*||||A||$, whence $||A|| \leq ||A^*||$, and we obtain the reverse inequality upon replacing A by A^*.

We have already encountered several examples of C^*-algebra.

Example 1.1.3 *Let H be a Hilbert space and $B(H)$ be the set of all bounded operators on H. Define sums and products of elements of $B(H)$ in the standard manner and equip this set with the operator norm*

$$||A|| = sup\{||Ah|| : h \in H, ||h|| = 1\}.$$

The Hilbert space adjoint operation (i.e If $A \in B(H)$, then A^* is the unique operator in $B(H)$ satisfying,

$$< Ah, k >=< h, A^*k > \text{ for all} h, k \in H$$

defines an involution on $B(H)$ and with respect to these operations and this norm $B(H)$ is a C^*-algebra.

$$\begin{aligned}||A||^2 &= sup\{< Ah, Ah >, h \in H, ||h|| = 1\}\\ &= sup\{< h, A^*Ah >, h \in H, ||h|| = 1\}\\ &\leq sup\{||A^*Ah||, h \in H, ||h|| = 1\}\\ &= ||A^*A|| \leq ||A^*||A|| = ||A||^2, then:\\ ||A||^2 &= ||AA^*||.\end{aligned}$$

Example 1.1.4 *Let X be a locally compact space and $C_0(X)$ be the set of all continuous complex-valued functions over X which vanish at infinity.*

By this we mean that for each $f \in C_0(X)$ and $\varepsilon > 0$ there is a compact $K \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \in X \backslash K$, the complement of K in X. Define the algebraic operations by $(f+g)(x) = f(x) + g(x), (\alpha f)(x) = \alpha f(x), (fg)(x) = f(x)g(x)$, and involution by $f^*(x) = \overline{f(x)}$. Finally, introduce a norm by

$$||f|| = sup\{|f(x)| : x \in X\}.$$

It follows that $C_0(X)$ is a commutative C^*-algebra. In particular, the norm identity is valid because

$$||ff^*|| = sup\{|f(x)\bar{f}(x)|, x \in X\} = sup\{|f(x)|^2, x \in X\} = ||f||^2.$$

Note that $C_0(X)$ has the identity if and only if X is compact.

Example 1.1.5 *The algebra $L^\infty(\mu)$ of all essentially bounded measurable functions (with pointwise algebraic operations and essential suppremum norm) associated with a measure space (S, μ) is a C^*-algebra;*

for $f \in L^\infty(\mu)$, $||f||_\infty = inf\{\alpha : |f(x)| \leq \alpha \; a.e.\}$, *and* $f^*(x) = f(x)^-$. *thus*

$$\begin{aligned} ||ff^*||_\infty &= inf\{\alpha : |f(x)f^-(x)| \leq \alpha \; a.e.\} \\ &= inf\{\alpha : |f(x)|^2 \leq \alpha \; a.e.\} \\ &= inf\{\alpha : |f(x)| \leq \alpha^{1/2} \; a.e.\} \\ &= inf\{\beta^2 : |f(x)| \leq \beta \; a.e.\} \\ &= (inf\{\beta : |f(x)| \leq \beta \; a.e.\})^2 \\ &= ||f||^2_\infty. \end{aligned}$$

For a simple example of C^*-algebra, let C with $||z|| = |z|$ and $z^* = \bar{z}$ (complex conjugate).

We refer to A^* as the adjoint of $A(\in \mathcal{U})$, and describe A as *self* $-$ *adjoint* if $A = A^*$, *normal* if $AA^* = A^*A$, *unitray* if $A^*A = AA^* = I$. The set of all self-adjoint elements of $\mathcal{U}$ is a real vector space.

Each A in $\mathcal{U}$ can be expressed (uniquely) in the form $H+iK$ where H and K are self-adjoint elements of $\mathcal{U}$, the "real" and "imaginary" parts of A; moreover, A is normal if and only if H and K commute. From (ii) of definition 1.1.7 , A is invertible if and only if A^* is invertible, and then $(A^{-1})^* = (A^*)^{-1}$.

If $\mathcal{U}$ and $\mathcal{B}$ are Banach algebras with involutions, a mapping φ from $\mathcal{U}$ into $\mathcal{B}$ is described as a $*$-homomorphism if it is a homomorphism (that is, it is linear, multiplicative, and carries the unit of $\mathcal{U}$ onto that of $\mathcal{B}$) with the additional property that $\varphi(A^*) = \varphi(A)^*$ for each A in $\mathcal{U}$. If further, φ is one-to-one, it is described as a $*$-isomorphism. If $\mathcal{U}$ is a Banach algebra with involution, a subset $\mathcal{F}$ of $\mathcal{U}$ is said to be self-adjoint if it contains the adjoint of each of its members. A self-adjoint subalgebra of $\mathcal{U}$ is termed a $*$-subalgebra.

1.2 Abelian C^*-algebras

Befor introducing this section we recall some notation. let $\mathcal{U}$ be a C^*-algebra without identity. If we wish to consider a property of $\mathcal{U}$ which deals with the identity element (*e.g.* the definition of the spectrum of $A \in \mathcal{U}$) the following theorem is quite useful for our purpose.

Theorem 1.2.1 *For each C^*-algebra $\mathcal{U}$ there exists a C^*-algebra $\hat{\mathcal{U}}$ with identity containing $\mathcal{U}$ as a closed ideal. If $\mathcal{U}$ has no identity then*

$$\frac{\hat{\mathcal{U}}}{\mathcal{U}} \cong \mathbf{C}$$

.

Proof. See [14; proposition 1.1.3]. □

Note that $\hat{\mathcal{U}}$ is called the C^*-algebra obtained from $\mathcal{U}$ by adjoining the identity 1.

Let $\mathcal{U}$ be a C^*-algebra with identity 1. Then the *spectrum* $sp_{\mathcal{U}}(A)$ of an element A in $\mathcal{U}$ is the set of all complex numbers α such that $A - \alpha I$ is not invertible.

If $\mathcal{U}$ is a C^*-algebra without identity, then the spectrum $sp_{\mathcal{U}}(A)$ of element A in $\mathcal{U}$ is the spectrum of A as an element of the C^*-algebra $\hat{\mathcal{U}}$ obtained from $\mathcal{U}$ by adjoining the identity 1.

By [8, Theorem 3.2.3] $sp_{\mathcal{U}}(A)$ is a nonempty closed subset of the closed disk in $\mathbf{C}$ with center 0 and radius $||A||$, and by [8, Theorem 4.1.5], $sp_{\mathcal{U}}(B) = sp_{\mathcal{B}}(B)$ for each C^*-subalgebra $\mathcal{B}$ of $\mathcal{U}$, where $B \in \mathcal{B}$. Then we can now omit the suffices $\mathcal{U}$ and $\mathcal{B}$, and denoted by $sp(B)$ the spectrum of B relative to either algebra. (In general case if $\mathcal{U}$ is a complex Banach algebra, $\mathcal{B}$ is a closed subalgebra that contains the unit I of $\mathcal{U}$, $B \in \mathcal{B}$, and $\alpha \in sp_{\mathcal{U}}(B)$, then $\alpha I - B$ has no inverse in $\mathcal{U}$; accordingly, it has no inverse in $\mathcal{B}$, so $\alpha \in sp_{\mathcal{B}}(B)$. Hence $sp_{\mathcal{U}}(B) \subseteq sp_{\mathcal{B}}(B)$, and Example 3.2.19 of [8] shows that strict inclusion can occur.).

We describe an element A of a C^*-algebra $\mathcal{U}$ as ***positive*** if A is self-adjoint and $sp(A) \subseteq \mathbf{R}^+$, we denote by $\mathcal{U}^+$ the set of all positive elements of $\mathcal{U}$. The concept of a positive element is the abstraction of the positive number (*i.e.*$\mathcal{U} = \mathbf{C}$ the condition $sp(z) \subseteq \mathbf{R}^+$ implies that $sp(z) = \{\alpha | \alpha 1 - z \text{ is not invertible }\} = \{z\} \subseteq \mathbf{R}^+$ then $z \geq 0$). If $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{U}$, a self-adjoint element B of $\mathcal{B}$ is positive relative to $\mathcal{B}$ if and only if it is positive relative to $\mathcal{U}$ (that is, $\mathcal{B}^+ = \mathcal{B} \cap \mathcal{U}^+$). Since it has the same spectrum in $\mathcal{B}$ as in $\mathcal{U}$.

Let M be a self-adjoint subspace of a C^*-algebra $\mathcal{U}$, and contains the unit I of $\mathcal{U}$. The set $M \cap \mathcal{U}^+$ of all positive elements of M is denoted by M^+. If $M \subseteq \mathcal{B} \subseteq \mathcal{U}$, where $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{U}$, then $M^+ = M \cap U^+ = M \cap \mathcal{B} \cap \mathcal{U}^+ = M \cap \mathcal{B}^+$; so M^+ is unchanged if M is viewed as a subspace of $\mathcal{B}$ instead of $\mathcal{U}$.

A linear functional ρ on M is said to be ***positive*** if $\rho(A) \geq 0$, for each A in M^+; if, further, $\rho(I) = 1, \rho$ is described as a *state* of M.

With H a Hilbert space and x in H, the equation $w_x(A) =< Ax, x >$ $(A \in B(H))$ defines a linear functional w_x on $B(H)$. In view of the equivalence of two concepts of positivity for Hilbert space operators, $w_x(A) \geq 0$ whenever $A \in B(H)^+$. Since, also, $w_x(I) = ||x||^2$, it follows that w_x is a positive linear functional on $B(H)$, and is a state if $||x|| = 1$. If $\mathcal{U}$ is a C^*-subalgebra of $B(H)$, and (as usual) M is a self-adjoint subspace of $\mathcal{U}$ that contains I, the restriction $w_x|_M$ is a positive linear functional on M. The states of M that arise in this way, from unit vectors in H, are termed *vector states*.

Theorem 1.2.2 *If M is a self-adjoint subspace of a C^*-algebra $\mathcal{U}$ and contains the unit I of $\mathcal{U}$, a linear functional ρ on M is positive if and only if ρ is bounded and $||\rho|| = \rho(I)$.*

Proof. See [8; Theorem 4.3.2]. □

From the above theorem each state of M is a bounded linear functional on M, with $||\rho|| = 1$. Accordingly, the set $\mathbf{S}(M)$ of all states of M is contained in the surface of the unit ball in the Banach dual space $M^{\neq}$. It is convex and wk^*-closed, since

$$\mathbf{S}(M) = \{\rho \in M^{\neq} : \rho(I) = 1, \rho(A) \geq 0 (A \in M^+)\},$$

by Alaoghu' s Theorem $ball(M^{\neq})$ is wk^*-compact and wk^*-topology is Hausdorff, then $\mathbf{S}(M)$ is wk^*-compact. It follows that $\mathbf{S}(M)$, with the