IN THE NAME OF ALLAH

MODULES WITH BOUNDED SPECTRA

BY

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TO:

The callous hands of my father
The separation endured my mother
and the pains they suffered together
that of which I am still unaware

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ABSTRACT

MODULES WITH BOUNDED SPECTRA

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Let R be a commutative ring with identity and let M be an R-module. We examine the sitnation where for each prime ideal p of R the set of all p-prime submodules of M is finite. In case R is Noetherian and M is finitely generated, we prove that this condition is equivalent to there eany authors (e.g., [3,6,19,24,25,28,30]). Our approach to unique extension, being a positive integer n such tht for every prime ideal p of R, the numver of p-prime submodules of M is less than or equal to n. We further show that in this case, there is at most one p-prime submodule for all but finitely many prime ideals p of R.

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CHAPTER 1
INTRODUCTION

Throughout this dissertation all rings are commutative with identity (except for a short section in 1.3), all modules are unitary and $I \lhd R$ means that I is an ideal of the ring R. Additionally, $N \leq M$ means that N is a submodule of the R-module M. In this chapter we prove some of the preliminary results which are used in other chapters.

1.1 Some Results on Rings and Modules

Definition. If M is an R-module and N a submodule of $M, (N : M)$ is defined to be $(N : M) = \{r \in R : rM \subseteq N\}$, obviously $(N : M)$ is an ideal of R.

Definition A proper submodule N of a module M over a ring R is said to be primary if $rb \in N$ for $r \in R$ and $b \in M$ implies that either $b \in N$ or $r^n M \subseteq N$ for some positive integer n.

Lemma 1.1.1 *Let M be a finitely generated R-module and N be a submodule of M. Then $\frac{M}{N}$ is a finitely generated R-module.*

Proof: It is clear that $\frac{M}{N}$ is an R-module. Suppose that $\{x_1, x_2, \cdots, x_n\}$ generate M, it is easy to see that $\{x_1 + N, x_2 + N, \cdots, x_n + N\}$ generate $\frac{M}{N}$.

Lemma 1.1.2 *Let M be an R-module and N be a submodule of M. Then $Ann(\frac{M}{N}) = (N : M)$.*

Proof: Let $r \in Ann(\frac{M}{N})$. Then $r(m+N) = N$ for each $m+N \in \frac{M}{N}$. That is $rm \in N$ for each $m \in M$. So $rM \subseteq N$. Therefore $r \in (N : M)$. Now, let $r \in (N : M)$. So $rM \subseteq N$, that is, $r(m+N) = N$ for each $m \in M$. Hence $r \in Ann(\frac{M}{N})$.

Lemma 1.1.3 *Let M be a finitely generated R-module and I be an ideal of R such that $\sqrt{I} = I$. Then $(IM : M) = I$ if and only if $Ann(M) \subseteq I$.*

Proof: The necessity is obvious. Assume that $Ann(M) \subseteq I$ and let r be an element in R which is contained in $(IM : M)$. If M is generated by n elements, then there exists $y \in I$ such that $r^n + y \in Ann(M)$, by [8, p. 50, Theorem 75]. Accordingly $r^n \in I$ and therefore $(IM : M) \subseteq \sqrt{I} = I$. Now, we easily see that $(IM : M) = I$.

Lemma 1.1.4 *If M is a finitely generated R-module and N is a submodule of M such that $(N : M) = p$, where p is a prime ideal of R, then $(p\frac{M}{N} : \frac{M}{N}) = p$.*

Proof: By Lemma 1.1.1 $\frac{M}{N}$ is a finitely generated R-module. By Lemma 1.1.2 $Ann(\frac{M}{N}) = p$. Then by Lemma 1.1.3, $(p\frac{M}{N} : \frac{M}{N}) = p$.

Suppose that S is a multiplicatively closed subset in R.

Lemma 1.1.5 *Let M be an R-module and N be a submodule of M. Then every element of $S^{-1}N$ has the form $\frac{b}{s}$, such that $b \in N$ and $s \in S$.*

Proof: We know that every element of $S^{-1}N$ has the form $\sum\limits_{i=1}^{n} \frac{r_i}{s_i} b_i$ where $r_i \in R, s_i \in S$ and $b_i \in N$. Let $s = \Pi_{i=1}^{n} s_i$ and $\overline{s_i} = \Pi_{\substack{j=1 \\ j \neq i}}^{n} s_j$. then we have

$$\sum_{i=1}^{n} \frac{r_i}{s_i} b_i = \frac{\sum_{i=1}^{n} r_i \overline{s_i} b_i}{s}$$

If we set $b = \sum_{i=1}^{n} r_i \overline{s_i}$, then $b \in N$ and $\sum_{i=1}^{n} \frac{r_i}{s_i} b_i = \frac{b}{s} \in S^{-1}N$.

Lemma 1.1.6 *Let M be an R-module and N be a submodule of M which is generated by $\{x_1, x_2, \cdots, x_n\}$. Then $S^{-1}N$ is an $S^{-1}R$-submodule of $S^{-1}M$ which is generated by $\{\frac{x_1}{1}, \frac{x_2}{1}, \cdots, \frac{x_n}{1}\}$.*

Proof: It is clear.

Lemma 1.1.7 *Let N and K be two submodules of an R-module M. Then*

i) $S^{-1}(N \cap K) = S^{-1}N \cap S^{-1}K$

ii) $S^{-1}(N : M) \subseteq (S^{-1}N : S^{-1}M)$

iii) $S^{-1}(N + K) = S^{-1}N + S^{-1}K$

iv) $S^{-1}M/S^{-1}N \simeq S^{-1}(M/N)$

Proof: See [2].

Lemma 1.1.8 *Let M be an R-module and N be a submodule of M. If $S^{-1}N \neq S^{-1}M$, then $(N : M) \cap S = \phi$.*

Proof: Suppose that $r \in (N : M) \cap S$. We know that every element of $S^{-1}M$ is of the form $\frac{m}{s}$ where $m \in M$ and $s \in S$. So let $\frac{m}{s} \in S^{-1}M$. Then $rs \in S$ and $rm \in N$. Hence $\frac{m}{s} = \frac{rm}{rs} \in S^{-1}N$. That is, $S^{-1}M \subseteq S^{-1}N$. Note that $S^{-1}N \subseteq S^{-1}M$. Hence $S^{-1}M = S^{-1}N$ which is a contradiction.

Lemma 1.1.9 *Let M be a finitely generated R-module such that $S^{-1}M = 0$. Then there exists an element $s \in S$ such that $sM = 0$.*

Proof: Let $M = Rm_1 + Rm_2 + \cdots + Rm_n$. Since $S^{-1}M = 0$, $\frac{m_1}{1} = \frac{m_2}{1} = \cdots = \frac{m_n}{1} = 0$. So we can find $t_1, t_2, \cdots, t_n \in S$ such that $t_1 m_1 = t_2 m_2 = \cdots = t_n m_n = 0$. If we set $s = t_1 t_2 \cdots t_n$, then we are done.

Lemma 1.1.10 *Let M be a finitely generated R-module and N be a submodule of M. Then $S^{-1}M \neq S^{-1}N$ if and only if $(N : M) \cap S = \phi$.*

Proof: First suppose that $(N : M) \cap S = \phi$, we show that $S^{-1}M \neq S^{-1}N$. If $S^{-1}M = S^{-1}N$, then $\frac{S^{-1}M}{S^{-1}N} = 0$ and by Lemma 1.1.7 $S^{-1}(M/N) = 0$. By Lemma 1.1.9 there exists an element $s \in S$ such that $s(\frac{M}{N}) = 0$. That is $s \in Ann(\frac{M}{N}) = (N : M)$. Then $s \in (N : M) \cap S$. That is, $(N : M) \cap S \neq \phi$ which is a contradiction so $S^{-1}M \neq S^{-1}N$. Conversely suppose that $S^{-1}M \neq S^{-1}N$ then by Lemma 1.1.8 $(N : M) \cap S = \phi$.

Lemma 1.1.11 *Let R be an integral domain and N be a primary submodule of the R-module M. If $S^{-1}M = S^{-1}N$, then $(N : M) \cap S \neq \phi$.*

Proof: Let $m \in M \backslash N$. So $\frac{m}{1} \in S^{-1}M = S^{-1}N$ and hence there exist $s \in S$ and $n \in N$ such that $\frac{m}{1} = \frac{n}{s}$. So there exists $t \in S$ such that $t(sm - n) = 0$. That is $tsm = tn \in N$ and since N is a primary submodule of M then there exists a positive integer k such that $(ts)^k M \subseteq N$, that is, $(ts)^k \in (N : M) \cap S$. So $(N : M) \cap S \neq \phi$.

Proposition 1.1.12 *Let M be an R-module and S be a multiplicatively closed subset of R. Then the $S^{-1}R$-modules $S^{-1}M$ and $S^{-1}R \otimes_R M$ are isomorphic, more precisely, there exists a unique isomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ which $f((\frac{a}{s}) \otimes m) = \frac{am}{s}$ for all $a \in R, m \in M$ and $s \in S$.*

Proof: See [2].

Proposition 1.1.13 *Let M be an R-module, R be a domain and K be its field of fractions of R. Then*

i) $K \otimes M \simeq KM$

ii) The torsion submodule of M, T is the kernel of the mapping f : $m \longmapsto 1 \otimes m$ of M into $K \otimes M$.

Proof: i) In the previous proposition let $S = R - \{0\}$.

ii) Note that f is an R-module homomorphism. Let $m \in Kerf$. Then $f(m) = 0$ so $1 \otimes m = 0$ and by (i) $\frac{m}{1} = 0$. So there exists $s \in R - \{0\}$ such that $sm = 0$. Thus $m \in T$, the torsion submodule of M.

Now let $m \in T$. Then there exists $r \in R - \{0\}$ such that $rm = 0$. So $\frac{rm}{1} = 0$. Then by (i) $1 \otimes rm = 0$. That is, $f(rm) = 0$. Since $f(m) \in K \otimes M$ and $K \otimes M$ is an K-module and $\frac{1}{r} \in K$ then $\frac{1}{r}.rf(m) = 0$, that is $f(m) = 0$. So $m \in kerf$. Hence $kerf = T$.

Now we prove some results on vector spaces which are used in the other chapters.

Lemma 1.1.14 *Let V be a finite-dimensional vector space over the finite field F. Then V is finite.*

Proof: Since V is a finite dimensional vector space, there exist $\beta_1, \beta_2, \cdots, \beta_n \in V$ such that $\{\beta_1, \beta_2, \cdots, \beta_n\}$ is a basis of V. For each $x \in V$ we can find $b_1, b_2, \cdots, b_n \in F$ such that

$$x = b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n.$$

Since F is finite, there exist finitely many choices for each b_i. So there exist finitely many choices for x and, that is, V is a finite vector space.

A consequence of Lemma 1.1.14 is that if M is a finitely generated module over a finite ring R, then M is finite.

Corollary 1.1.15 *Every finite-dimensional vector space over a finite field has finitely many subspaces.*

Proof: It is clear.

Lemma 1.1.16 *Let F be a field and V be a vector space over F with linearly independent elements x, y. Then $\{F(x+cy) : c \in F\}$ is a family of one-dimensional subspaces of V such that $F(x+cy) \neq F(x+dy)$ for all $c \neq d$ in F.*

Proof: First we show that $F(x+cy)$ is a subspace of V for each $c \in F$. Let $\alpha = f_1(x+cy)$ and $\beta = f_2(x+cy)$ be two elements of $F(x+cy)$ where $f_1, f_2 \in F$. Then for each $g \in F, \alpha + g\beta = f_1(x+cy) + gf_2(x+cy) = (f_1+gf_2)(x+cy) \in F(x+cy)$. Now we show that $F(x+cy) \neq F(x+dy)$ for all $c \neq d$ in F. If $F(x+cy) = F(x+dy)$, then since $x+cy \in F(x+cy), x+cy \in F(x+dy)$. So there exists $a \in F$ such that $x+cy = a(x+dy) = ax+ady$. That is, $(1-a)x+(c-ad)y = 0$. Since x and y are linearly independent elements of V, then $a = 1$ and $d = c$ which is a contradiction. Hence $F(x+cy) \neq F(x+dy)$.

Corollary 1.1.17 *Let V be a vector space over a field F, then V has not any non-zero proper subspace if and only if $dimV = 1$.*

Proof: First suppose that $dimV = 1$ and W is a proper subspace of V. Then $dimW < 1$. That is $dimW = 0$ and so $W = 0$.

Conversely suppose that V has not any non-zero proper subspace. We show that $dimV = 1$. If $dimV \neq 1$, then by Lemma 1.1.16 V has a family of one-dimensional subspaces and it is a contradiction.

Corollary 1.1.18 *Let V be a finite-dimensional vector space over an* *infinite field F. Then V has some infinite subspaces if and only if $dimV > 1$.*

Proof: It follows by Lemma 1.1.16.

Definition A ring R is said to be uniformly infinite if R/p is an infinite field for every maximal ideal p of R.

For example $F[x]$ is a uniformly infinite ring for every infinite field F.

Definition The module M is called a multiplication module provided that for each submodule N of M there exists an ideal I of R such that $N = IM$.

Lemma 1.1.19 *An R-module M is a multiplication module if and only if for each element m in M there exists an ideal I of R such that $Rm = IM$.*

Proof: The necessity is clear.

Conversely, suppose that for each element $m \in M$ there exists an ideal I such that $Rm = IM$. Let N be a submodule of M. For each $x \in N$ there exists an ideal I_x such that $Rx = I_xM$. Let $I = \sum_{x \in N} I_x$. Then $N = IM$. It follows that M is a multiplication module.

Lemma 1.1.20 *Every cyclic R-module is multiplication.*

Proof: Let M be a cyclic R-module. Obviously $M \simeq R/I$ for some ideal I of R. So for every submodule N of M, we have $N \simeq J/I$ for some ideal J containing I. Since $J/I = J(R/I)$, we have $N = JM$.

Lemma 1.1.21 *(Nakayama) If J is an ideal of R, then the following conditions are equivalent.*

i) J is contained in every maximal ideal of R.

ii) $1 - j$ is a unit for every $j \in J$.

iii) If M is a finitely generated R-module such that $JM = M$, then $M = 0$.

iv) If N is a submodule of a finitely generated R-module such that $M = JM + N$, then $M = N$.

Proof: See[2].

Let R be a local ring, m its maximal ideal, $k = R/m$, its residue field. Let M be a finitely generated R-module. M/mM is annihilated by m and hence is naturally an $\frac{R}{m}$-module, that is a k-vector space.

Lemma 1.1.22 *Let $x_i (1 \leq i \leq n)$ be the elements of M whose images in $\frac{M}{mM}$ form a basis of this vector space. Then $x_i^{\prime}s$ generate M.*

Proof: See[2].

Lemma 1.1.23 *Let p be a prime ideal of R and M be a cyclic R-module. Then M_p is cyclic.*

Proof: Suppose that $M =< m >$. Let $\frac{x}{s} \in M_p$. Then there exists $r \in R$ such that $x = rm$. So $\frac{x}{s} = \frac{rm}{s} = \frac{r}{s}.\frac{m}{1}$. Hence M_p is a cyclic R_p-module which is generated by $\frac{m}{1}$.

Lemma 1.1.24 *let $a_1, a_2, \cdots, a_n$ be ideals of R and let p be a prime ideal containing $\cap a_i$. Then $a_i \subseteq p$ for some $i \in \{1, 2, 3, \cdots, n\}$.*

Proof: See[2].

Proposition 1.1.25 *Let p be a prime ideal of R and $a_1, a, \cdots, a_n$ be ideals of R. Then the following are equivalent.*

i) $\cap a_i \subseteq p$

ii) $\Pi_i a_i \subseteq p$

iii) $\exists j \in \{1, 2, \cdots, n\}, a_j \subseteq p$

Proof: See[17].

Definition An ideal a is said to be irreducible if $a = b \cap c$, where b and c are ideals of R then either $a = b$ or $a = c$.

Lemma 1.1.26 *In a Noetherian ring R, every ideal is a finite intersection of irreducible ideals.*

Proof: Suppose not, then the set of ideals in R for which the lemma is false is not empty and hence has a maximal element a. Since a is reducible, we have $a = b \cap c$ where $b \supset a$ and $c \supset a$. Hence each of b and c is a finite intersection of irreducible ideals and therefore so is a, which is a contradiction.

Lemma 1.1.27 *In a Noetherian ring, every irreducible ideal is primary.*

Proof: See[2].

Lemma 1.1.28 *In a Noetherian ring R, every ideal has a primary decomposition.*

Proof: See[2].

Lemma 1.1.29 *In a Noetherian ring R, every ideal I contains a power of its radical.*

Proof: See[2].

Lemma 1.1.30 *Let M be a finite R-module and let $a = Ann(M)$. Then $\frac{R}{a}$ is a finite ring.*

Proof: Suppose that $M = \{m_1, m_2, \cdots, m_k\}$. Then $a = Ann(m_1) \cap Ann(m_2) \cap \cdots \cap Ann(m_k) = \bigcap_{i=1}^{k} Ann(m_i)$. Suppose that $\frac{R}{a}$ is an infinite ring. We know that the kernel of the homomorphism f of R in to $\frac{R}{Ann(M_1)} \oplus \frac{R}{Ann(m_2)} \oplus \cdots \oplus \frac{R}{Ann(m_k)}$ is $a = Ann(M)$. So $\frac{R}{a}$ is embedded in $\frac{R}{Ann(m_1)} \oplus \cdots \oplus \frac{R}{Ann(m_k)}$. Therefore, there exists $i \in \{1, 2, \cdots, n\}$ such that $\frac{R}{Ann(m_i)}$ is infinite. But $\frac{R}{Ann(m_i)} \simeq Rm_i$ and since Rm_i is a submodule of the finite module M, this is a contradiction.

Lemma 1.1.31 *Let a be an ideal of the Noetherian ring R such that $\frac{R}{p}$ is a finite ring for each prime ideal $p \supseteq a$. Then $\frac{R}{a}$ is a finite ring.*

Proof: We know that in a Noetherian ring R, every ideal has a primary decomposition. So there exist primary ideals $Q_i (1 \leq i \leq n)$ such that $a = \bigcap_{i=1}^{n} Q_i$. Suppose that Q_i is a p_i-primary. Then $a = \bigcap_{i=1}^{n} Q_i \subseteq \bigcap_{i=1}^{n} \sqrt{Q_i} = \bigcap_{i=1}^{n} p_i$, where $p_i^{\prime}s (1 \leq i \leq n)$ are prime ideals of R. By lemma 1.1.29 the ideal a contains a power of its radical. Suppose that $(\sqrt{a})^m \subseteq a$ where $m \in Z$, then $(\bigcap_{i=1}^{n} p_i)^m \subseteq a$ so $(\prod_{i=1}^{n} p_i)^m \subseteq a$ and that is $\prod_{i=1}^{n} p_i^m \subseteq a \subseteq \bigcap_{i=1}^{n} p_i$. Without lose of generality we can assume that $q_1 q_2 \cdots q_k \subseteq a \subseteq q_1 \cap q_2 \cap \cdots \cap q_k$ where q_i is prime ideal for each $i \in \{1, 2, \cdots, k\}$.

Now consider the chain $R \supseteq q_1 \supseteq q_1 q_2 \supseteq q_1 q_2 q_3 \supseteq \cdots \supseteq q_1 q_2 \cdots q_k$ and note that $(q_1 q_2 \cdots q_{i-1})/(q_1 q_2 \cdots q_i)$ is a finitely generated $\frac{R}{q_i}$-module for each $1 \leq i \leq k$. Since $\frac{R}{q_i}$ is a finite field, $(q_1 q_2, \cdots, q_{i-1})/(q_1 q_2 \cdots q_i)$ is a finite $\frac{R}{q_i}$-module. Since $\frac{R}{q_1}$ and $\frac{q_1}{q_1 q_2}$ are finite and $\frac{R}{q_1} \simeq (R/q_1 q_2)/(q_1/q_1 q_2)$, $\frac{R}{q_1 q_2}$ is finite because

$$|\frac{R}{q_1 q_2}| = |\frac{R/q_1 q_2}{q_1/q_1 q_2}| \times |\frac{R}{q_1}|.$$