IN THE NAME OF GOD

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Faculty of Sciences

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ON THE COMMUTANT AND NORM OF CERTAIN MULTIPLICATION OPERATORS AND COMPOSITION OPERATORS ON CERTAIN BANACH SPACE OF ANALYTIC FUNCTIONS

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To my wife

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ABSTRACT

ON THE COMMUTANT AND NORM OF CERTAIN MULTIPLICATION OPERATORS AND COMPOSITION OPERATORS ON CERTAIN BANACH SPACE OF ANALYTIC FUNCTIONS

By

Mahmood Haji Shaabani

In the first chapter, we give some background information including definitions of various spaces of analytic functions and properties of certain operators. We will also some theorems and examples needed for the chapters that follow.

In chapter II, we obtain the norm of a composition operator C_φ on the Hardy space H^2, whenever φ is a linear fractional mapping of the form $\varphi(z)=\dfrac{1}{cz+d}$. In fact extend some results obtained by P.Bourdon, E. Fry, C. Hammond and C. Spofford ([4]).

In chapter III, we obtain a representation for the norm of certain compact weighted composition operator $C_{\psi,\varphi}$ on the Hardy space H^2, whenever $\varphi(z)=az+b$ and $\psi(z)=az-b$. We also estimate the norm and the essential norm of a class of noncompact weighted composition operators under certain conditions on φ and ψ. Moreover, we determine the exact values of the norm and the essential norm of such operators in a special case. In chapter IV, we investigate which Toeplitz operators, Hankel operators and multiplication operators commute with a given composition operator on $H^2(\beta)$ such that

$\varphi: D \to D$ is an analytic map, which is neither an elliptic disc automorphism of finite periodicity nor the identify map. We prove that the commutant of a simple composition operator is actually the strong operator closure of the polynomials in C_φ. Also, when S is a bounded linear operator in the commutant of C_φ, we show that under certain conditions, S is a polynomial in C_φ.

In chapter V, we investigate which Toeplitz operators and multiplication operators commute with a given composition operator C_φ on A_α^p such that $\varphi: D \to D$ is an analytic map which is neither an elliptic disc automorphism of finite periodicity nor the identity map. Let S be a bounded linear operator in the commutant of C_φ. We show that under certain conditions, S is a polynomial in C_φ.

In chapter VI, we investigate the commutant of the operator M_φ for certain function $\varphi \in M(B)$. In particular, when φ is a polynomial or a rational function with poles off $\overline{G}$, under certain conditions on the coefficients, we show that $\{M_\varphi\}' = \{M_z\}'$. In [17], $\check{Z}.\check{C}u\check{c}kovic'$ and Dashan Fan have shown that if $G = \{z \in C : T < |z| < 1\}$, $B = L_a^2(G)$ *and* $p(z) = z + a_2 z^2 + \ldots + a_n z^n$, where $a_i \geq 0$ and $p(z) - p(1)$ has n distinct zeros, then $\{M_p\}' = \{M_\psi : \psi \in M(B)\}$. As a result, Theorem 6.2.7 extends the result obtained in [17] to Banach space of analytic functions on various domains G and certain polynomial or rational symbols. Also we extend the results obtained in [30]. Moreover, in both papers the authors used the condition that zeros of the function p outside $\overline{G}$ are distinct which we omit this condition.

Contents

Chapter 1

PRELIMINARIES

1 PRELIMINARIES

1.1 Preliminaries

Definition 1.1.1 Let D denote the open unit disk in the complex plane. The Hardy space H^2 is the space of all analytic functions defined on D, whose Taylor coefficients, in the expansion about the origin, are square summable. Also we recall that H^∞ is the space of all bounded analytic functions defined on D. For $\alpha \in D$, the reproducing kernel at α for H^2 is defined by $K_\alpha(z) = \frac{1}{1-\bar{\alpha}z}$. An easy computation shows that $< f, K_\alpha >= f(\alpha)$, whenever $f \in H^2$. For more information about these spaces one can see [18].

Definition 1.1.2 Let dA be the normalized area measure on D. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space $A^p_\alpha = A^p_\alpha(D)$ of the disk is the space of analytic functions in $L^p(D, dA_\alpha)$, where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z).$$

If f is in $L^p(D, dA_\alpha)$ we write

$$\|f\|_{p.\alpha} = [\int_D |f(z)|^p dA_\alpha(z)]^{\frac{1}{p}}.$$

When $1 \leq p < \infty$, the space $L^p(D, dA_\alpha)$ is a Banach space and the weighted Bergman space A^p_α is closed in $L^p(D, dA_\alpha)$, so A^p_α is a Banach space. Let $L^\infty(D)$ denote the space of essentially bounded functions on D. For $f \in L^\infty(D)$, we define

$$\|f\|_\infty = \text{esssup}\{|f(z)| : z \in D\}.$$

The space $L^\infty(D)$ is a Banach space with the above norm. As usual, let H^∞ denote the space of bounded analytic functions on D. It is clear that H^∞ is closed in $L^\infty(D)$ and hence is a Banach space. For more information about these spaces one can see [27].

Definition 1.1.3 Let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$. We consider the space of sequences $f = \{\hat{f}(n)\}$ such that

$$\|f\|^2 = \|f\|_\beta^2 = \sum |\hat{f}(n)|^2[\beta(n)]^2 < \infty.$$

We shall use the notation

$$f(z) = \sum \hat{f}(n)z^n \qquad (z \in D). \tag{1.1}$$

If n ranges only over the nonnegative integers, these are formal power series, otherwise they are formal Laurent series. We shall denote these spaces by $H^2(\beta)$ (power series case) and $L^2(\beta)$ (Laurent series case). $H^2(\beta)$ is called a weighted Hardy space. These notations are suggested by the notations in the classical case $\beta = 1$. When we do not wish to distinguish between the two cases, we shall refer to the space as H. These spaces are Hilbert spaces with the inner product

$$< f, g >= \sum \hat{f}(n)\hat{g}(n)[\beta(n)]^2$$

for every f and g in H. Let $\hat{f}_k(n) = \delta_{nk}$, for $n, k \in \mathbb{Z}$, in the notation of equation (1.1) we have $f_k(z) = z^k$. Then f_k is an orthogonal basis. It is clear that $\|f_k\| = \beta(k)$.

We recall that $H^\infty(\beta)$ denotes the set of formal power series $\phi(z) = \sum \hat{\phi}(n)z^n$ $(n \geq 0)$ such that $\phi H^2(\beta) \subseteq H^2(\beta)$ and $L^\infty(\beta)$ denotes the set of formal Laurent series $\phi(z) = \sum \hat{\phi}(n) z^n$ $(-\infty < n < \infty)$ such that $\phi L^2(\beta) \subseteq L^2(\beta)$. Since f_0 is in both $L^2(\beta)$ and $H^2(\beta)$, and since $\phi f_0 = \phi$ for all Laurent series ϕ, we see that

$$L^\infty(\beta) \subseteq L^2(\beta) \quad , \quad H^\infty(\beta) \subseteq H^2(\beta).$$

Note that for any formal Laurent series ϕ and each integer m, we have

$$z^m \phi(z) = \sum \hat{\phi}(k) z^{k+m} = \sum \hat{\phi}(k-m) z^k.$$

Thus, for every ϕ in $L^2(\beta)$ or $H^2(\beta)$,

$$< f_m \phi, f_n >= \hat{\phi}(n-m)[\beta(n)]^2. \tag{1.2}$$

Let $w \in D$ and let λ_w be the linear functional of point evaluation at w; that is, $\lambda_w(f) = f(w)$ for every f in $H^2(\beta)$. By [45], λ_w is a bounded linear functional and by Riesz representation Theorem

$$\lambda_w(f) =< f, K_w >,$$

where

$$K_w(z) = \sum_{j=0}^{\infty} \frac{\overline{w}^j z^j}{\beta(j)^2}.$$

Moreover,

$$\|K_w\|^2 = \sum_{j=0}^{\infty} \frac{(|w|^2)^j}{\beta(j)^2}.$$

For examples of weighted Hardy spaces, let A be the normalized Lebesgue area measure on D. For $\alpha > -1$, the weighted Dirichlet space D_α is the collection of all functions f, holomorphic in D, for which the complex derivative f' belongs to A^2_α. The space D_α is a Hilbert space with the norm $\|.\|_{D_\alpha}$, defined by

$$\|f\|_{D_\alpha}^2 = |f(0)|^2 + \int_D |f'|^2 dA_\alpha.$$

A standard power series computation using integration in polar coordinates and the orthogonality of the monomials z^n in $L^2(\partial D)$ shows that a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ holomorphic in D, belongs to A_α^2 if and only if

$$\sum_{n=0}^{\infty} (n+1)^{-1-\alpha} |a_n|^2 < \infty,$$

and belongs to D_α if and only if

$$\sum_{n=0}^{\infty} (n+1)^{1-\alpha} |a_n|^2 < \infty.$$

A complex valued function φ defined on D is called a multiplier of D_α if $\varphi D_\alpha \subset D_\alpha$ and the collection of all these multipliers is denoted by $\mathcal{M}(D_\alpha)$. Note that $D_1 = H^2$ and for $\alpha > -1$, $D_{\alpha+2} = A_\alpha^2$ and, in addition, if $-1 < \alpha < 1$, D_α is a subset of H^2. For $\alpha \geq 1$, $\mathcal{M}(D_\alpha) = H^\infty$. For each analytic map $\varphi : D \to D$, the composition operator C_φ on D_α is defined by $C_\varphi(f) = f \circ \varphi$. Let φ be an analytic self-map of the unit disk and $\alpha \geq 1$. Then automatically $\{\varphi^{[n]}, n \in \mathbb{N}\}$ is bounded in D_α and C_φ is a bounded operator on D_α, where $\varphi^{[n]}$ denotes the n times composition of φ with itself. If $\alpha < 1$, $\mathcal{M}(D_\alpha) \subset H^\infty$, $\mathcal{M}(D_\alpha) \neq H^\infty$ and if $\varphi : D \to D$ is an analytic map, it is not obvious that C_φ should take D_α into itself and there are actually many examples for which it does not, also this is not necessary that $\varphi \in D_\alpha$ or $\{\varphi^{[n]}, n \in \mathbb{N}\}$ is bounded in D_α (see [34]).

Let T be a bounded operator on a Hilbert space H. We recall that $\|T\|_e$, the essential norm of T, is the norm of its equivalence class in the Calkin algebra. Since the spectral radius of the operator T^*T equals $\|T^*T\| = \|T\|^2$, we study the spectrum of T^*T when trying to determine $\|T\|$. We say that the operator T is norm-attaining, if there is a nonzero $h \in H$ such that $\|T(h)\| = \|T\|\|h\|$.

The following propositions have shown, further connection between $\|T\|$ and the spectrum of T^*T.

Proposition 1.1.4 *Let h be an element of Hilbert space H. Then $\|T(h)\| = \|T\|\|h\|$ if and only if $T^*T(h) = \|T\|^2 h$.*

Proof. See, [24].□

Proposition 1.1.5 *If $\|T\|_e < \|T\|$, then T attains its norm at an element of H.*

Proof. See, [24].□

Theorem 1.1.6 *Suppose that $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional transformation mapping D into itself, where $ad - bc \neq 0$. Then $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$ maps D into itself, $g(z) = \frac{1}{-\bar{b}z+\bar{d}}$ and $h(z) = cz + d$ are in H^∞, and the adjoint of C_φ on H^2 has the form*

$$C_\varphi^* = T_g C_\sigma T_h^*.$$

For the definition of T_f, see Definition 1.1.8.

Proof. See, [13].□

Definition 1.1.7 Let $P_\alpha : L^p(D, dA_\alpha) \longrightarrow A^p_\alpha$, $1 < p < \infty$, be the Bergman projection. Then P_α is an integral operator represented by

$$P_\alpha g(z) = \int_D K(z,w)g(w)dA_\alpha(w),$$

where

$$\begin{aligned} K(z,w) &= \frac{1}{(1-z\overline{w})^{2+\alpha}} \\ &= \Sigma_{n=0}^{\infty}\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}(z\overline{w})^n. \end{aligned}$$

Given a function $f \in L^\infty(D)$, we define an operator T_f on A^p_α, $(1 < p < \infty)$ by

$$T_f(g) = P_\alpha(fg).$$

T_f is called the Toeplitz operator on the weighted Bergman space with symbol f. If we define $M_f : L^p(D, dA_\alpha) \longrightarrow L^p(D, dA_\alpha)$ by $M_f(g) = fg$, it is obvious that M_f is bounded. Since the Bergman projection for $1 < p < \infty$ is bounded (see [27]), we conclude that T_f is a bounded operator.

Definition 1.1.8 Let $P : L^2(\beta) \longrightarrow L^2(\beta)$ be the orthogonal projection onto $H^2(\beta)$. By (1.2), if $g(z) = \sum_{-\infty}^{\infty} \hat{g}(n)z^n$, then

$$\begin{aligned} Pg(z) &= < Pg, K_z > \\ &= < g, PK_z > \\ &= < g, K_z > \\ &= < g, \sum_{n=0}^{\infty} \frac{\overline{z}^n w^n}{\beta(n)^2} > \\ &= \sum_{n=0}^{\infty} \hat{g}(n) z^n. \end{aligned}$$

Given a function $f \in L^\infty(\beta)$, we define an operator T_f on $H^2(\beta)$ by

$$T_f(g) = P(fg).$$

T_f is called the Toeplitz operator on the weighted Hardy space with symbol f. If we define $M_f : L^2(\beta) \longrightarrow L^2(\beta)$ by $M_f(g) = fg$, then by [45], M_f is bounded. Since an orthogonal projection has norm 1, clearly T_f is bounded.

Lemma 1.1.9 *A sequence in $H^2(\beta)$ converges weakly if and only if it is bounded and converges pointwise.*

Proof. see [15].□

Definition 1.1.10 A linear fractional transformation is a mapping of the form

$$T(z) = \frac{az+b}{cz+d},$$

subject to the further condition $ad - bc \neq 0$ which is necessary and sufficient for T to be non-constant. We denote the set of all such maps by $LFT(\hat{\mathbb{C}})$. Our interest here is in $LFT(D)$, the subgroup of $LFT(\hat{\mathbb{C}})$ consisting of selfmaps of the unit disk D. Those that takes D onto itself are called automorphisms. Consideration of normal forms quickly shows that:

(a) Parabolic members of $LFT(D)$ have their fixed point on ∂D.

(b) Hyperbolic members of $LFT(D)$ must have attractive fixed point in $\overline{D}$, with the other fixed point outside D, and lying on ∂D if and only if the map is an automorphism of D.

(c) Loxodromic and elliptic members of $LFT(D)$ have a fixed point in D and a fixed point outside $\overline{\partial D}$. The elliptic ones are precisely the automorphism in $LFT(D)$ with this fixed point configuration. For more information about these spaces one can see [41].

Theorem 1.1.11 *(Denjoy-Wolff) If φ is not the identity and not an elliptic automorphism of D, is an analytic map of D into D, then there is a point a in $\overline{D}$, so that the iterates $\varphi^{[n]}$ of φ converge to a uniformly on compact subsets of D.*

Proof. See [15].□

Lemma 1.1.12 *Let $\varphi : D \to D$ be an analytic map which is not an elliptic disc automorphism, and let $\{\varphi^{[n]}, n \in \mathbb{N}\}$ be bounded in $H^2(\beta)$. Let a be the Denjoy-Wolff point of φ. Then for each fixed positive integer l, $(\varphi^{[n]})^l$ converges to a^l weakly in $H^2(\beta)$.*

proof. By Theorem 1.1.10, $(\varphi^{[n]})^l$ converges to a^l pointwise and by hypothesis $(\varphi^{[n]})^l$ is bounded in the $H^2(\beta)$ norm. Hence by Lemma 1.1.9, the proof is complete.□

In following Theorems 1.1.14 and 1.1.15, C_φ is defined on H^2.

Theorem 1.1.13 *Suppose that $\varphi(z) = \frac{b}{cz+d}$ is a self-map of D, then $\|C_\varphi\| > \|C_\varphi\|_e$.*

Proof. See [4, Corollary 3.10].□

Theorem 1.1.14 *Suppose that $\varphi(z) = \frac{b}{d-z}$, where $0 < b < d-1$ and $\lambda = \|C_\varphi\|^2$. Then λ is the unique positive number satisfying*

$$\sum_{k=0}^{\infty} a_k \left(\frac{1}{\lambda}\right)^{k+1} = 1,$$

where for $k \geq 0$,

$$a_k = \frac{b^{2k}}{d^{2k} + (-1)^k b^{2k} + (-1)^k \sum_{m=1}^{k-1} (-1)^m d^{2m} \sum_{j=0}^{k-m} C_j^{m+j} C_{k-m-j}^{k-1-j} b^{2j}}.$$

Proof. See [4, Proposition 4.4].□

Theorem 1.1.15 *(Mergelyan's Theorem) Suppose K is a compact set in the plane whose complement is connected. If f is a continuous complex function on K which is analytic in the interior of K, and if $\epsilon > 0$, then there is a polynomial p such that $|p(z) - f(z)| < \epsilon$ for all z in K.*

Proof. See [40, Theorem 20.5].□