In the name of God

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A Goal Programming Approach to Fuzzy Linear Regression

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Abstract

Many researches have been carried out in fuzzy linear regression since the past three decades. In this thesis, a brief literature review of previous works is presented, and some of their shortcomings are pointed out. Then, three types of fuzzy linear regression models are considered to fit to a set of experimental fuzzy/crisp input and fuzzy output data. To construct the regression models, some simple goal programming models are proposed. The proposed models take into account the centers of fuzzy data as an important feature as well as their spreads. Furthermore, the models can deal with both symmetric and non-symmetric data. To show the efficiency of the proposed models, they are compared with some earlier methods based on simulation studies and numerical examples according to two criteria of goodness.

Contents

Introduction

Regression analysis is a powerful statistical methodology to analyze the phenomena in which one variable named output (response or dependent) variable depends on one or more variables named input (explanatory or independent) variables. In regression, the main question is that "how one can predict the value of a dependent variable based on some observations which have been obtained from some experiments".

In the classical regression analysis, deviations between the observed and estimated values are supposed to be caused by observations errors or variables which are not included in the model. However, in the fuzzy regression analysis, deviations are assumed to depend on the fuzziness of the system structure. In fact, some phenomena in the real world depend on uncertain factors. So, they cannot be exactly analyzed. Fuzzy regression analysis is a tool to handle such phenomena. The essential question is: **"How can one predict a dependent variable based on some observations (experimental data) in a fuzzy environment?"**

Fuzzy regression analysis is a relatively new field of research. Since Tanaka et al. [48] initiated research on fuzzy linear regression (FLR) analysis, this area has been widely developed in theory and application. One approach to deal with FLR is linear programming (LP) approach, which was first introduced by Tanaka et al. [48]. They proposed a linear programming method to fit a fuzzy linear regression model to a set of given data. In their model, it is assumed that the inputs are crisp (real) numbers, however, the observed responses are triangular fuzzy numbers. Their method then was developed by others (e.g. see [17, 18, 36, 37, 40, 41, 44, 46, 47, 48]). Another approach is least-squares approach, which was first introduced by Celmins [3] and developed by others (see e.g. [8, 9, 22, 23, 30, 31, 32, 52, 56]). Some authors discussed features, advantages and shortcomings of different methods and some of them tried to rectify the shortcomings (e.g. [8, 17, 18, 20, 23, 38]). A review on previous researches in more details is presented in Chapter 2.

The main purpose behind this thesis is constructing the fuzzy linear regression model for the problems in which the input data are crisp or fuzzy, and the output data are fuzzy numbers. To this end, the goal programming approach is used. The thesis contains two parts.

In Part I, containing chapters 1, 2, and 3, some basic miscellaneous contexts are introduced. In Chapter 1, some preliminaries containing fuzzy set theory and goal programming are presented. In Chapter 2, the classical and fuzzy regression analysis are introduced and some available regression methods are reviewed and discussed. Furthermore, a shortcoming of the least-squares method introduced by Kim and Bishu [23] is illustrated and two modifications of this method are proposed. Finally, two criteria of goodness are presented which are used in subsequent chapters to compare different methods.

In Part II, containing chapters 3-5, the goal programming approach is proposed to construct different fuzzy linear regression models.

Part I

Miscellaneous Basic Contexts

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, some preliminaries containing the foundation of fuzzy set theory, a brief review on fuzzy set theory, and goal programming which are required in subsequent chapters are presented. In Section 1.2, the fuzzy set theory is introduced briefly, and goal programming is introduced in Section 1.3.

1.2 Fuzzy set theory

1.2.1 Foundation

Most of the real-world phenomena depend on uncertain factors which cannot be described exactly. Most of human inferences are not consistent with the two-valued logic. When we say: "a person is tall or is truthful", when we say: "the sky is cloudy", when we evaluate the students by their scores, and in an infinite number of such expressions and situations, we do not use the two-valued logic. In the first and second examples we use infinite-valued logic and in the third example, we use e.g. the 81-valued logic (when the scores are 0, 0.25, 0.5, ..., 20). Obviously, using the classical logic to describe such situations is not suitable. What can we say when the sky is neither completely cloudy nor completely sunny, e.g. when it is 70% or 10% cloudy? Is the statement "The sky is cloudy" true or false? How can the two-valued logic and its deductions (inferences) describe, analyze, and solve the real-world problems? How can one formulate the real-world systems by crisp relations? The real world is full of problems having uncertainty or ambiguity. Therefore, a tool is needed to describe uncertainty or ambiguity in the real world phenomena.

In 1965, L.A. Zadeh [57] introduced a theory whose objects - fuzzy sets - are sets with boundaries that are not precise. The membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of degree. When A is a fuzzy set and x is a relevant object, the proposition "x is a member of A" is not necessarily either true or false, as required by two-valued logic, but it may be true only to some degree, the degree to which x is actually a member of A. The theory of fuzzy sets then was applied to a wide variety of fields such as operations research, management science, artificial intelligence, control theory, and statistics.

1.2.2 Fuzzy sets and fuzzy numbers

A fuzzy set, first introduced by Zadeh [57] is defined as follows:

Definition 1.2.1. (Fuzzy set)

Let X denote a universal set. Then a fuzzy subset $\tilde{A}$ of X is defined by its membership function:

$$\tilde{A} : X \to [0,1], \tag{1.2.1}$$

which assigns to each element $x \in X$ a real number $\tilde{A}(x)$ where the value of $\tilde{A}(x)$ represents the degree of membership of x in $\tilde{A}$.

A fuzzy subset $\tilde{A}$ can be characterized as a set of ordered pairs of element x and grade $\tilde{A}(x)$:

$$\tilde{A} = \{(x, \tilde{A}(x)) | x \in X\}. \tag{1.2.2}$$

An ordinary set A is expressed by its characteristic function $\chi_A : X \to \{0,1\}$ as:

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin X \end{cases} \tag{1.2.3}$$

Therefore, it can be interpreted as a special fuzzy set whose membership function contains only two values 0 and 1. So its membership function $\tilde{A}(x)$ is identical to its characteristic function $\chi_A(x)$.

The concept of α-level set of a fuzzy set has an important role in fuzzy set theory.

Definition 1.2.2. (α-level set) [39]

For $\alpha \in [0,1]$, the α-level set of a fuzzy set $\tilde{A}$ is the crisp set

$$\tilde{A}_\alpha = \{x \in X | \tilde{A}(x) \geq \alpha\}. \tag{1.2.4}$$

Some other concepts of fuzzy sets are defined as follows:

Support: [39] The support of a fuzzy set $\tilde{A}$ on X, denoted by $supp(\tilde{A})$, is the set of points in X at which $\tilde{A}(x) > 0$, i.e.,

$$supp(\tilde{A}) = \{x \in X | \tilde{A}(x) > 0\}. \tag{1.2.5}$$

Height: [39] The height of a fuzzy set $\tilde{A}$ on X, denoted by $hgt(\tilde{A})$, is the least upper bound of $\tilde{A}(x)$, i.e.,

$$hgt(\tilde{A}) = \sup_{x \in X} \tilde{A}(x). \tag{1.2.6}$$

Normality: [39] The fuzzy set $\tilde{A}$ on X is said to be normal if its height is unity, i.e., if there is $x \in X$ such that $\tilde{A}(x) = 1$. If a fuzzy set is not normal, it is said to be subnormal.

Two set-theoretic operations on fuzzy sets originally proposed by Zadeh are as follows:

Equality: The fuzzy sets $\tilde{A}$ and $\tilde{B}$ on X are equal, denoted by $\tilde{A} = \tilde{B}$, if and only if their membership functions are equal everywhere on X:

$$\tilde{A} = \tilde{B} \Leftrightarrow \tilde{A}(x) = \tilde{B}(x) \; \forall x \in X. \tag{1.2.7}$$

Containment: [39] The fuzzy set $\tilde{A}$ is contained in $\tilde{B}$ (or a subset of $\tilde{B}$), denoted by $\tilde{A} \subseteq \tilde{B}$, if and only if its membership function is less than or equal to that of $\tilde{B}$ everywhere on X:

$$\tilde{A} \subseteq \tilde{B} \Leftrightarrow \tilde{A}(x) \leq \tilde{B}(x) \; \forall x \in X. \tag{1.2.8}$$

The notion of convexity of fuzzy sets has an important role in fuzzy optimization. A fuzzy set $\tilde{A}$ of $\mathbb{R}^n$ is said to be convex if for any $\alpha \in [0, 1]$, $\tilde{A}_\alpha$ is a convex set.

A special kind of fuzzy sets is fuzzy number. A fuzzy number is a convex normal fuzzy set of the real line $X = \mathbb{R}^1$ whose membership function is piecewise continuous.

A fuzzy number $\tilde{A}$ is called positive (non-negative), denoted by $\tilde{A} > 0$ (≥ 0), if its membership function $\tilde{A}(x)$ satisfies $\tilde{A}(x) = 0, \forall x \leq 0 (< 0)$. A negative (non-positive) fuzzy number is defined similarly.

Algebraic operations on real numbers can be extended to similar ones on fuzzy sets. To this end, the extension principle proposed by Zadeh [57] is used.

Definition 1.2.3. [39] The n-dimensional fuzzy cartesian product of the fuzzy sets $\tilde{A}_i$ in X_i, for $i = 1, \cdots, n$, denoted by $\tilde{A}_1 \times \cdots \times \tilde{A}_n$, is defined as a fuzzy set in $X_1 \times \cdots \times X_n$ whose membership function is

$$\tilde{A}_1 \times \cdots \times \tilde{A}_n(x_1, \cdots, x_n) = \min\{\tilde{A}_1(x_1), \cdots, \tilde{A}_n(x_n)\}. \tag{1.2.9}$$

Definition 1.2.4. (Extension principle) [39]
Let $f : X \to Y$ be a mapping from a set X to a set Y. Then, for each fuzzy set $\tilde{A}$ in X the fuzzy set $\tilde{B}$ in Y is induced by f as follows:

$$\tilde{B} = \{(y, \tilde{B}(y)) | y = f(x), x \in X\} \tag{1.2.10}$$

with

$$\tilde{B}(y) = \begin{cases} \sup_{y=f(x), x\in X} \tilde{A}(x) & f^{-1}(y) \neq \emptyset \\ 0 & f^{-1}(y) = \emptyset \end{cases} \qquad \text{for all } y \in Y \tag{1.2.11}$$

where, $f^{-1}(y)$ is the inverse image of y.

Now, by setting $X = \mathbb{R}^2$ and $Y = \mathbb{R}$, the addition, subtraction, and multiplication of two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are obtained as follows:

Addition:

$$(\tilde{A} + \tilde{B})(x) = \sup_{x=a+b} \min\{\tilde{A}(a), \tilde{B}(b)\} = \sup_{a\in\mathbb{R}^1} \min\{\tilde{A}(a), \tilde{B}(x-a)\}\} \qquad (1.2.12)$$

Subtraction:

$$(\tilde{A} - \tilde{B})(x) = \sup_{x=a-b} \min\{\tilde{A}(a), \tilde{B}(b)\} = \sup_{a\in\mathbb{R}^1} \min\{\tilde{A}(a), \tilde{B}(a-x)\} \qquad (1.2.13)$$

Multiplication:

$$(\tilde{A} \times \tilde{B})(x) = \sup_{x=ab} \min\{\tilde{A}(a), \tilde{B}(b)\} \qquad (1.2.14)$$

Hereafter, we denote the multiplication $\tilde{A} \times \tilde{B}$ by $\tilde{A}\tilde{B}$, for simplicity. By using the extension principle, the following proposition can be proved for fuzzy numbers.

Proposition 1.2.1. *Suppose r and s are real numbers and $\tilde{A}$ and $\tilde{B}$ are fuzzy numbers. Then*

$$(r+s)\tilde{A} \subseteq r\tilde{A} + s\tilde{A}, \qquad (1.2.15)$$

$$(\tilde{A} + \tilde{B})\tilde{C} \subseteq \tilde{A}\tilde{C} + \tilde{B}\tilde{C}. \qquad (1.2.16)$$

Proof. To prove (1.2.15) we have to show that:

$$((r+s)\tilde{A})(x) \leq (r\tilde{A} + s\tilde{A})(x) \quad \forall x \in \mathbb{R}. \qquad (1.2.17)$$

Let us denote the membership functions of the real numbers r and s by their characteristic functions χ_r and χ_s, respectively. By using (1.2.12) and (1.2.14) we have:

$$\begin{aligned}((r+s)\tilde{A})(x) &= \sup_{uv=x}\{\inf\{(\chi_r+\chi_s)(u), \tilde{A}(v)\}\} \\ &= \sup_{uv=x}\{\inf\{\sup_{y+z=u}\{\inf\{\chi_r(y), \chi_s(z)\}\}, \tilde{A}(v)\}\} \\ &= \sup_{uv=x,\ r+s=u}\{\inf\{\chi_r(r), \chi_s(s), \tilde{A}(v)\}\} \\ &= \sup_{(r+s)v=x}\{\tilde{A}(v)\} \\ &= \sup_{(r+s)v=x}\{\inf\{\tilde{A}(v), \tilde{A}(v)\}\}.\end{aligned} \tag{1.2.18}$$

On the other hand:

$$\begin{aligned}(r\tilde{A}+s\tilde{A})(x) &= \sup_{y+z=x}\{\inf\{(\chi_r\tilde{A})(y), (\chi_s\tilde{A}(z)\}\} \\ &= \sup_{y+z=x}\{\inf\{\sup_{tu=y}\{\inf\{\chi_r(t), \tilde{A}(u)\}\}, \\ &\qquad \sup_{wv=z}\{\inf\{\chi_s(w), \tilde{A}(v)\}\}\}\} \\ &= \sup_{y+z=x,\ ru=y,\ sv=z}\{\inf\{\chi_r(r), \tilde{A}(u), \chi_s(s), \tilde{A}(v)\}\} \\ &= \sup_{ru+sv=x}\{\inf\{\tilde{A}(u), \tilde{A}(v)\}\}.\end{aligned} \tag{1.2.19}$$

The Inequality (1.2.17) is obtained from (1.2.18) and (1.2.19) because

$$\{(v,v)|v\in\mathbb{R}\}\subseteq\{(u,v)|u,v\in\mathbb{R}\}.$$

□

To prove (1.2.16) see [35].

One important kind of fuzzy numbers is L-R fuzzy numbers which are defined as follows.

Definition 1.2.5. (L-R fuzzy number) [39]

A fuzzy number $\tilde{A}$ is said to be an L-R fuzzy number if

$$\tilde{A}(x) = \begin{cases} L\left(\frac{a-x}{\alpha}\right) & x \leq a, \quad \alpha > 0 \\ R\left(\frac{x-a}{\beta}\right) & x \geq a, \quad \beta > 0 \end{cases} \tag{1.2.20}$$

where a is the mean value of $\tilde{A}$ and α and β are left and right spreads, respectively, and L(.) is a left shape function satisfying

1. L(x)=L(-x)

2. L(0)=1

3. L(x) is nonincreasing on [0,∞).

R(.) is a right shape function, which is defined similar to L(.).

Such an L-R fuzzy number is denoted by $\tilde{A} = (a, \alpha, \beta)_{LR}$, by using its mean value, left and right spreads, and shape functions.

Remark 1.2.1. When the spreads of the L-R fuzzy number $\tilde{A} = (a, \alpha, \beta)_{LR}$ are zero, $\tilde{A}$ is called a degenerated L-R fuzzy number and identically denoted by its mean value: $\tilde{A} = (a, 0, 0)_{LR} = a$.

Dubois and Prade [13] showed the following exact formulas for addition and subtraction and approximate formulas for multiplication of two L-R fuzzy numbers.

Addition: If $\tilde{A} = (a, \alpha, \beta)_{LR}$ and $\tilde{B} = (b, \gamma, \delta)_{LR}$ then

$$(\tilde{A} + \tilde{B}) = (a + b, \alpha + \gamma, \beta + \delta)_{LR}. \tag{1.2.21}$$

Subtraction: If $\tilde{A} = (a, \alpha, \beta)_{LR}$ and $\tilde{B} = (b, \gamma, \delta)_{RL}$ then

$$\tilde{A} - \tilde{B} = (a - b, \alpha + \delta, \beta + \gamma)_{LR}. \tag{1.2.22}$$

Multiplication: If $\tilde{A} = (a, \alpha, \beta)_{LR}$ and $\tilde{B} = (b, \gamma, \delta)_{LR}$ and the spreads are small enough compared with the mean values then

$$\tilde{A}\tilde{B} \simeq \begin{cases} (ab, a\gamma + b\alpha, a\delta + b\beta)_{LR} & if \quad \tilde{A} > 0, \tilde{B} > 0, \\ (ab, b\alpha - a\delta, b\beta - a\gamma)_{RL} & if \quad \tilde{A} < 0, \tilde{B} > 0, \\ (ab, -b\beta - a\delta, -b\alpha - a\gamma)_{RL} & if \quad \tilde{A} < 0, \tilde{B} < 0. \end{cases} \tag{1.2.23}$$

Also, multiplication of the L-R fuzzy number $\tilde{A} = (a, \alpha, \beta)_{LR}$ by the scalar $\lambda \in \mathbb{R}$ can be obtained as follows:

$$\lambda\tilde{A} = \begin{cases} (\lambda a, \lambda\alpha, \lambda\beta)_{LR} & \lambda \geq 0 \\ (\lambda a, -\lambda\beta, -\lambda\alpha)_{RL} & \lambda < 0. \end{cases} \tag{1.2.24}$$

Triangular fuzzy number: Set $L(x) = R(x) = 1-|x|$, if $0 \leq x \leq 1$ and 0 elsewhere, then the L-R fuzzy number $\tilde{A} = (a, \alpha, \beta)_{LR}$ is called a triangular fuzzy number and denoted by $\tilde{A} = (a, \alpha, \beta)$ (Figure 1.1). If $\alpha = \beta$ then $\tilde{A}$ is called a symmetric triangular fuzzy number and denoted by $\tilde{A} = (a, \alpha)$. The set of all triangular fuzzy numbers (containing the real numbers) is denoted by $T(\mathbb{R})$. The membership function of the triangular fuzzy number $\tilde{A} = (a, \alpha, \beta)$ is as follows:

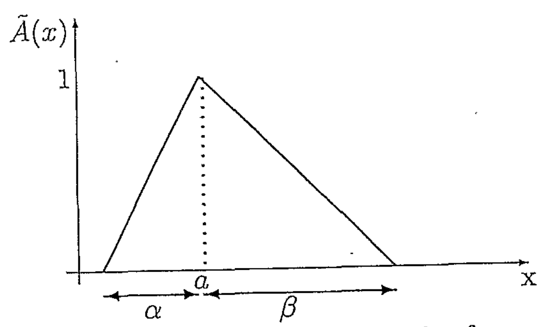

Figure 1.1: The membership function of the triangular fuzzy number $\tilde{A} = (a, \alpha, \beta)$