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Stable Topology and Pure Filters on BL-Algebras

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DEDICATED TO:

MY LATE FATHER,

MY WIFE, MY DAUGHTER, MY MOTHER,

MY BROTHERS AND MY SISTER

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Abstract

L. P. Belluce and S. Sessa studied stable topology and pure ideals in the framework of MV-algebras.

In 1998 Peter Hájek introduced the variety of BL-algebras and showed that the variety of MV-algebras actually is a subvariety of the variety of BL-algebras. Thus it makes sense to generalize the notion of stable topology to BL-algebras. But in fact since the multiplication ($\odot$) is a fundamental operation and filters are basic notions in BL-algebras defined in terms of $\odot$, we prefer to present stable topology based on filters. We also prove some more theorems regarding different properties of this topology on BL-algebras.

This thesis consists of three sections. In the first section we state and review notions of MV-algebra, ideals, prime spectrum, stable topology and pure ideals for MV-algebras. In the second section we recall the definition of a BL-algebra A, a filter F, a deductive system and $Spec(A)$ with more preliminary facts that we need in the sequel. In the third section we define $F-topology$ which is actually the same as spectral topology but in terms of filters and introduce the stable topology on $Spec(A)$. We show that the topological space $Spec(A)$ with the stable topology is compact but not T_0.

We define pure filters of A and prove some important results. In fact let $Max(A)$ be the set of all maximal filters of A. We consider the topology induced by $F-topology$ on $Max(A)$ and show that $F-topology$ and stable topology coincide on subspace $Max(A)$. We show that pure filters of A are in one to one correspondence with closed subsets of $Max(A)$. We also investigate some conditions for purity of a filter F by considering $\sigma(F) = \{a \in A | \; y \wedge z = 0 \text{ for some } z \in F \text{ and } y \in a^{\perp}\}$ and stability of $U(F)$ where $U(F)$ is an open set in $Spec(A)$ with F-topology.

Contents

Chapter 1

MV-algebras

1.1 Introduction

The theory of MV-algebras has its origin in the study of the system of infinite-valued logic originated by Lukasiewicz. The completeness of the propositional Łukasiewicz logic was first published by Rose and Rosser in 1958. An earlier proof by Wajsberg was never published.

In 1958 C. C. Chang developed an algebraic version of Łukasiewicz propositional logic and provided an algebraic proof of the completeness. The resulting algebraic system became known as an MV-algebra. MV-algebras, therefore, stand in relation to the Łukasiewicz infinite valued logic as Boolean algebras stand in relation to classical 2-valued logic. Boolean algebras, of course, have not stayed glued to their origin in logic, their uses showing up in other areas of mathematics. Moreover there has been extensive investigations concerning their structures.

1.2 Preliminaries

Definition 1.2.1. [5] *An MV-algebra is a algebra* $(A, \oplus, -, 0)$ *with a binary operation* $\oplus : A \times A \longrightarrow A$*, a unary operation* $- : A \longrightarrow A$ *and a constant 0 satisfying the following equations for each* $x, y, z \in A$*:*

(1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(2) $x \oplus y = y \oplus x$;

(3) $x \oplus 0 = x$;

(4) $\overline{\overline{x}} = x$;

(5) $x \oplus \overline{0} = \overline{0}$;

(6) $\overline{(\overline{x} \oplus y)} \oplus y = \overline{(\overline{y} \oplus x)} \oplus x$.

We note the axioms 1-3 state that $(A, \oplus, 0)$ is an abelian monoid.

Example 1.2.2. [5] *(i) A singelton* $\{0\}$ *is a trivial example of an MV-algebra.*

(ii) Consider the real unit interval $[0, 1]$*, and for all* $x, y \in [0, 1]$*, let* $x \oplus y =_{def} min(1, x + y)$ *and* $\overline{x} =_{def} 1 - x$*. It is easy to see that* $[0, 1] = ([0, 1], \oplus, -, 0)$ *is an MV-algebra.*

(iii) Given an MV-algebra A and a set X, the set A^X *of all functions* $f : X \longrightarrow A$ *becomes an MV-algebra if the operations* $+, -$ *and the element* 0 *are defined pointwise.*

Remark 1.2.3. [5] *If A is an MV-algebra, then we can define the binary operations* $\odot, \wedge, \vee, \longrightarrow, \ominus$ *and the constant* 1 *as follow :*

(1) $a \odot b =_{def} \overline{\overline{a} \oplus \overline{b}}$;

(2) $a \wedge b = (a \oplus \overline{b}) \odot b$;

(3) $a \vee b = (a \odot \overline{b}) \oplus b$;

(4) $a \longrightarrow b = \overline{a} \oplus b$;

(5) $x \ominus y =_{def} x \odot \overline{y}$;

(6) $1 =_{def} \overline{0}$.

Corollary 1.2.4. [5] *The following identities are immediate consequence of* $\overline{\overline{x}} = x$.

(1) $\overline{1} = 0$,

(2) $x \oplus y = \overline{(\overline{x} \odot \overline{y})}$,

(3) $x \oplus 1 = 1$,

(4) $(x \ominus y) \oplus y = (y \ominus x) \oplus x$.

Proof: (1) $\overline{1} = \overline{\overline{0}} = 0$.

(2) By Remark 1.2.3 (1), $\overline{x} \odot \overline{y} = \overline{(\overline{\overline{x}} \oplus \overline{\overline{y}})} = \overline{(x \oplus y)}$. Therefore we have $x \oplus y = \overline{(\overline{x} \odot \overline{y})}$.

(3) From (2), it is trivial since $1 = \overline{0}$,

(4)

$$\begin{aligned} (x \ominus y) \oplus y &= (x \odot \overline{y}) \oplus y \\ &= \overline{(\overline{x} \oplus \overline{\overline{y}})} \oplus y \\ &= \overline{(\overline{x} \oplus y)} \oplus y \\ &= \overline{(\overline{y} \oplus x)} \oplus x \\ &= (y \ominus x) \oplus x. \end{aligned}$$

Lemma 1.2.5. [5] *Let A be an MV-algebra and* $x, y \in A$. *Then the following conditions are equivalent:*

(1) $\overline{x} \oplus y = 1$;

(2) $x \odot \overline{y} = 0$;

(3) $y = x \oplus (y \ominus x)$;

(4) There is an element $z \in A$ *such that* $x \oplus z = y$.

Proof: $(1 \Longrightarrow 2)$ $x \odot y = \overline{\overline{x} \oplus \overline{y}}$ then we have $x \odot \overline{y} = \overline{\overline{x} \oplus \overline{\overline{y}}} = \overline{\overline{x} \oplus y} = \overline{1} = 0$.

$(2 \Longrightarrow 3)$ From Corollary 1.2.4, $x \oplus (y \ominus x) = y \oplus (x \ominus y) = y \oplus (x \odot \overline{y}) = y \oplus 0 = y$.

$(3 \Longrightarrow 4)$ Take $z = y \ominus x$ then, $x \oplus (y \ominus x) = y$.

$(4 \Longrightarrow 1)$ Since $x \oplus \overline{x} = 1$, then $\overline{x} \oplus y = \overline{x} \oplus x \oplus z = 1 \oplus z = 1$.

Remark 1.2.6. [5] *Let A be an MV-algebra. For any two elements x and y of A let us agree to write* $x \leq y$ *iff x and y satisfy the equivalent conditions in Lemma 1.2.5. It follows that "* $\leq$ *" is a partial order, called the natural order of A. Indeed, reflexivity is equivalent to* $x \oplus \overline{x} = 1$*, antisymmetry follows from conditions 2 and 3, and transitivity follows from condition 4.*

1.3 MV-chain

Definition 1.3.1. [5] *An MV-algebra whose natural order is total is called a MV-chain.*

Therefore, by Lemma 1.2.5 (4), the natural order of the MV-chain $[0, 1]$ coincides with the natural order of the real numbers.

Lemma 1.3.2. [5] *Let A be an MV-algebra. For each* $a \in A$, $\overline{a}$ *is the unique solution x of the simultaneous equations:*

(i) $a \oplus x = 1$

(ii) $a \odot x = 0$

Proof: By Lemma 1.2.5, those two equations amount to writing $\overline{a} \leq x \leq \overline{a}$.

Lemma 1.3.3. [5] *In every MV-algebra A the natural order "$\leq$" has the following properties:*

1) $x \leq y$ *iff* $\overline{y} \leq \overline{x}$;

2) If $x \leq y$ *then for each* $z \in A$, $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$;

3) $x \odot y \leq z$ *iff* $x \leq \overline{y} \oplus z$.

Proof: (1) Suppose that $x \leq y$. Then by Remark 1.2.6, $1 = \overline{x} \oplus y = \overline{\overline{y}} \oplus \overline{x}$. This means that $\overline{y} \leq \overline{x}$. The converse is similar.

(2) Let $x \leq y$. Then $\overline{x} \oplus y = 1$ and by Lemma 1.2.5, there is an element z in A such that $x \oplus z = y$. Consider $\overline{x \oplus z} \oplus y \oplus z = \overline{y} \oplus y \oplus z = 1 \oplus z = 1$. Thus $x \oplus z \leq y \oplus z$. From (1), the other part is trivial.

(3) Consider $x \odot y \leq z$ iff $\overline{x \odot y} \oplus z = 1$ iff $\overline{x} \oplus \overline{y} \oplus z = 1$. This equivalent with $x \leq \overline{y} \oplus z$.

We recall that a poset L is a lattice if for each $x, y \in L$, $x \vee y$ and $x \wedge y$ exist in L.

Proposition 1.3.4. [5] *On each MV-algebra A the natural order determines a lattice structure. Specifically, the join* $x \vee y$ *and the meet* $x \wedge y$ *of the elements* x *and* y *as follow are given.*

(1) $x \vee y = (x \odot \overline{y}) \oplus y = (x \ominus y) \oplus y$;

(2) $x \wedge y = \overline{(\overline{x} \vee \overline{y})} = x \odot (\overline{x} \oplus y)$;

(3) $(x \ominus y) \wedge (y \ominus x) = 0$.

Proof: (1) Since $x \oplus \overline{x} = 1$ and $(x \ominus y) \oplus y = (y \ominus x) \oplus x$, from Lemma 1.3.3 (2) we have $x \leq (x \ominus y) \oplus y$ and $y \leq (x \ominus y) \oplus y$.

Suppose that $x \leq z$ and $y \leq z$. By Lemma 1.2.5 (1),(3), $\overline{x} \oplus z = 1$ and $z = (z \ominus y) \oplus y$. Then we can write

$$\begin{aligned}
\overline{((x \ominus y) \oplus y)} \oplus z &= (\overline{(x \ominus y) \ominus y}) \oplus y \oplus (z \ominus y) \\
&= (y \ominus \overline{(x \ominus y)}) \oplus \overline{(x \ominus y)} \oplus (z \ominus y) \\
&= (y \ominus \overline{(x \ominus y)}) \oplus \overline{x} \oplus y \oplus (z \ominus y) \\
&= (y \ominus \overline{(x \ominus y)}) \oplus \overline{x} \oplus z = 1.
\end{aligned}$$

It follows that $(x \ominus y) \oplus y \leq z$ which completes the proof of (1).

(2) We now immediately obtain (2) as a consequence of (1) and Lemma 1.3.3 (1).

$$\begin{aligned}
(3)\ (x \ominus y) \wedge (y \ominus x) = \;& (x \ominus y) \odot (\overline{(x \ominus y)} \oplus (y \ominus x)) \\
=\;& x \odot \overline{y} \odot (y \oplus \overline{x} \oplus (y \ominus x)) \\
=\;& x \odot (\overline{x} \oplus (y \ominus x)) \odot (\overline{(\overline{x} \oplus (y \ominus x))} \oplus \overline{y}) \\
=\;& (y \ominus x) \odot (\overline{(y \ominus x)} \oplus x) \odot (\overline{(\overline{x} \oplus (y \ominus x))} \oplus \overline{y}) \\
=\;& y \odot \overline{x} \odot (\overline{(y \ominus x)} \oplus x) \odot ((x \odot \overline{(y \ominus x)}) \oplus \overline{y}) \\
=\;& \overline{x} \odot (x \oplus \overline{(y \ominus x)}) \odot y \odot (\overline{y} \oplus (x \odot (\overline{y} \oplus x))) \\
=\;& \overline{x} \odot (x \oplus \overline{(y \ominus x)}) \odot (x \odot (\overline{y} \oplus x)) \odot (\overline{(x \odot (\overline{y} \oplus x))} \oplus y) \\
=\;& 0,
\end{aligned}$$

because, in each MV-algebra we have $x \odot \overline{x} = \overline{(\overline{x} \oplus \overline{\overline{x}})} = \overline{1} = 0$

1.4 Lattice Reduct

Definition 1.4.1. [5] *Let A be an MV-algebra and $x, y \in A$. We set $x \leq y$ iff $x \vee y = y$ or, equivalently, $x \wedge y = x$.*

It is easy to see that the relation $\leq$ is an ordering over A.

Proposition 1.4.2. [5] *Let A be an MV-algebra. Then the poset $(A, \leq)$ is a lattice such that, for every $x, y \in A$, $g.l.b\{x, y\} = x \wedge y$ and $l.u.b\{x, y\} = x \vee y$, where g.l.b and l.u.b means the greater lower bound and the least upper bound of x, y.*

Proof: $x \wedge y$ is a lower bound for x and y, because, $x \wedge (x \wedge y) = (x \wedge x) \wedge y = x \wedge y$ and $y \wedge (x \wedge y) = y \wedge x$. Suppose that $c \leq x$ and $c \leq y$. Then, $c = c \wedge c \leq x \wedge y$. Analogously $x \leq x \wedge y$ and $y \leq x \wedge y$. Suppose that $x \leq d$ and $y \leq d$. Then $x \vee y \leq d \vee d = d$.

Theorem 1.4.3. [5] *Let A be an MV-algebra. Then $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice.*

Proof: Obviously, A is bounded. So we need to show the distributivity law, that is

$$\begin{aligned} x \wedge (y \vee z) &= x \odot (\overline{x} \oplus (y \vee z)) \\ &= x \odot ((\overline{x} \oplus y) \vee (\overline{x} \oplus z)) \\ &= (x \odot (\overline{x} \oplus y)) \vee (x \odot (\overline{x} \oplus z)) \\ &= (x \wedge y) \vee (x \wedge z). \end{aligned}$$

Notice that the natural order makes every MV-algebra A in to a lattice with minimum element 0 and maximum element 1. We shall denote this lattice by $L(A)$ and called the lattice reduct of A.

1.5 Residuation

A residuated lattice ([14]) is a structure $(L, \vee, \wedge, \longrightarrow, \odot, 0, 1)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice, $(L, \odot, 1)$ is a commutative monoid and adjunction holds, i.e. for any $a, b, c \in L$, we have $a \leq b \longrightarrow c$ iff $a \odot b \leq c$. Now, let A be an MV-algebra. From Remark 1.2.3, we set $x \odot y = \overline{(\overline{x} \oplus \overline{y})}$ and $x \longrightarrow y = \overline{x} \oplus y$ then we conclude that $(L(A), \longrightarrow, \odot, \vee, \wedge, 0, 1)$ is a residuated lattice, because, let $a \leq b \longrightarrow c$, i.e. $a \leq \overline{b} \oplus c$, then $a \odot b \leq b \odot (\overline{b} \oplus c)$. Hence, $a \odot b \leq b \wedge c \leq c$. Viceversa, assume $a \odot b \leq c$, then $(a \odot b) \oplus \overline{b} \leq \overline{b} \oplus c$ and $a \leq a \vee \overline{b} \leq b \longrightarrow c$.

Theorem 1.5.1. [14] *A residuated lattice $(L, \vee, \wedge, \longrightarrow, \odot, 0, 1)$ is an MV-algebra iff it satisfies the additional condition : $(x \longrightarrow y) \longrightarrow y = (y \longrightarrow x) \longrightarrow x$, for any $x, y \in A$.*

Example 1.5.2. *We give an example of a finite residuated lattice which is an MV-algebra. Let $A = \{0, a, b, c, d, 1\}$, with $0 < a < b < 1$, $0 < c < d < 1$, but a, c and respective b,d are incomparable. We define*

$\longrightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

It is easy to see that $\overline{0} = 1$, $\overline{a} = d$, $\overline{b} = c$, $\overline{c} = b$, $\overline{d} = a$ and for all $x, y \in A$, we have $(x \longrightarrow y) \longrightarrow y = (y \longrightarrow x) \longrightarrow x$. For instance $(a \longrightarrow d) \longrightarrow d = 1 \longrightarrow d = d$ and $(d \longrightarrow a) \longrightarrow a = b \longrightarrow a = d$.

Example 1.5.3. *We give an example of a finite residuate lattice* $A = \{0, a, b, c, d, e, f, g\}$ *which is an MV-algebra but not an MV-chain, with* $0 < a < b < e < 1$, $0 < c < f < g < 1$, $a < d < g$, $c < d < e$, *but* $\{a, c\}, \{b, d\}, \{d, f\}, \{b, f\}$ *and, respective* $\{e, g\}$ *are incomparable. We define*

$\rightarrow$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	g	1	1	g	1	1	g	1	1
b	f	g	1	f	g	1	f	g	1
c	e	e	e	1	1	1	1	1	1
d	d	e	e	g	1	1	g	1	1
e	c	d	e	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	a	b	b	d	e	e	g	1	1
1	0	a	b	c	d	e	f	g	1

$\odot$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	0	0	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	0	0	0	c	c	c
d	0	0	a	0	0	a	c	c	d
e	0	a	b	0	a	b	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	0	a	c	c	d	f	f	g
1	0	a	b	c	d	e	f	g	1

and so A become a residuated lattice and from Theorem 1.5.1, it is easy to see that A is an MV-algebra.

Remark 1.5.4. [5] *Let A be an MV-algebra. For each x in A, we let $0x = 0$ and for each integer $n \geq 0$, $(n+1)x = nx \oplus x$.*

Proposition 1.5.5. [5] *Let x, y be elements of an MV-algebra A. If $x \wedge y = 0$ then for each integer n, $nx \wedge ny = 0$*

Proof: We know that for any $x, y \in A$, $x \geq y$ iff $y \leq x$. Therefore, if $x \wedge y = 0$, since in each MV-algebra we have $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$ then by Lemma 1.3.3, $x = x \oplus (x \wedge y) = (x \oplus x) \wedge (x \oplus y) \geq 2x \wedge y$, whence $0 = x \wedge y \geq 2x \wedge y$. It follows that $0 = 2x \wedge 2y = 4x \wedge 4y = \ldots$. The desired conclusion now follows from $nx \wedge ny \leq 2^n x \wedge 2^n y = 0$.

1.6 Homomorphim

Definition 1.6.1. [5] *Let A and B be MV-algebras. A function* $f : A \longrightarrow B$ *is a homomorphism iff it satisfies the following conditions, for each* $x, y \in A$:

(i) $f(0) = 0$,

(ii) $f(x \oplus y) = f(x) \oplus f(y)$,

(iii) $f(\overline{x}) = \overline{f(x)}$.

The kernel of $f : A \longrightarrow B$ is the set $Ker(f) = \{x \in A |\ f(x) = 0\} = f^{-1}(0)$.

1.7 Complemented elements of $L(A)$

Definition 1.7.1. [14] *A lattice A is said to be Boolean algebra , if it is distributive in which with every* $a \in A$ *there is associated an element* $\overline{a}$ *such that* $a \wedge \overline{a} = 0$ *and* $a \vee \overline{a} = 1$*. In this case,* $\overline{a}$ *is the complement of A.*

When L is distributive each $z \in L$ has at most one complement, denoted $\overline{z}$. We know that if $x \vee y = 1$ and $x \wedge y = 0$, then $y = \overline{x}$. Thus any complemented element x of $L(A)$ has $\overline{x}$ as complement. We further let $B(L)$ be the set of all complemented elements of the distributive lattice L. Note that 0 and 1 are elements of $B(L)$, because $(\overline{0}) = 1$ and $(\overline{1}) = 0$. As a matter of fact, $B(L)$ is a sublattice of L which is also a Boolean algebra. For any MV-algebra A we shall write $B(A)$ as an abbreviation of $B(L(A))$. Elements of $B(A)$ are called the Boolean elements of A. Therefore we have

Proposition 1.7.2. [5] *Let A be an MV-algebra and* $S(A) = \{x \in A | x \oplus x = x\} = \{x \in A | x \odot x = x\}$*. Then we have* *(i) x is complemented in $L(A)$ iff $x \oplus x = x$ iff $x \odot x = x$;*

(ii) $(S(A), \oplus, -, 0, 1)$ is a subalgebra of A;

(iii) $(S(A), \vee, \wedge, -, 0, 1)$ is a complemented sublattice of $L(A)$. Moreover, we have $x \vee y = x \oplus y$, for every $x, y \in S(A)$;

(iv) $(S(A), \oplus, \odot, -, 0, 1)$ is a Boolean algebra. It is the greatest Boolean subalgebra of A.

1.8 The ideals of MV-algebras

Definition 1.8.1. [5] *An ideal of an MV-algebra A is a subset I of A satisfying the following conditions:*

(i) $0 \in I$;

(ii) If $x \in I$, $y \in A$ and $y \leq x$ then $y \in I$;

(iii) If $x \in I$ and $y \in I$ then $x \oplus y \in I$.

Remark 1.8.2. [5] *The intersection of any family of ideals of an MV-algebra A is still an ideal of A. For every subset $W \subseteq A$, the intersection of all ideals I, which is containing W, is said to be the ideal generated by W and will be denoted $< W >$.*

It is easy to see that if $W = \emptyset$, then $< W >= \{0\}$. If $W \neq \emptyset$, then $< W >= \{x \in A \mid x \leq w_1 \oplus w_2 \oplus \ldots . \oplus w_k \ for\ some\ w_1, \ldots, w_k \in W\}$ ([5]). In particular, for each element z of MV-algebra A, the ideal $< z >=< \{z\} >$ is called the principal ideal generated by z, and we have $< z >= \{x \in A | \ nz \geq x \ for\ some\ integer\ n \geq 0\}$. Note that $< 0 >= \{0\}$ and $< 1 >= A$.