بسم الله الرحمن الرحیم

IN THE NAME OF GOD

Compactifications and Function Spaces on Weighted Semigroups

By:

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PREFACE

The most distinct and beautiful statements of any truth must take at least the mathematical form. we might so simplify the rules of moral philosophy as well as of arithmetic, that one formula would express them both.

H.D. Thoreau

Harmonic analysis is primarily the study of functions and measures on the topologico-algebraic structures.

The study of almost periodic functions (resp. weakly almost periodic and left uniformly continuous functions), has received a considerable attention during the past few decades (see [**3**] - [10], [**18**],[**29**],[**30**], and also the study of compactifications [**22**],[**26**],[**36**],[**37**]).

In recent years a number of authors have studied various sorts of functions and measures on weighted semigroup ([1],[**2**],[11],[**16**],[**34**]).

The general themes on which this thesis is based, are the study of these famous function spaces, and compactifications in the weighted case.

In [11] the spaces $\mathcal{AP}(S,w)$ and $\mathcal{WAP}(S,w)$ of weighted almost periodic and weakly almost periodic functions on a semigroup S are introduced. [**33**] gives a modification of these definitions. The new definitions which we follow are justified by the fact that we are able to establish necessary and sufficient conditions for the identites $\mathcal{AP}(S,w) = C(S,w)$ and $\mathcal{WAP}(S,w) = C(S,w)$ (and $\mathcal{WAP}(S,w) = \mathcal{L}_\infty(S,w)$ if and only if $\mathcal{L}_1(S,w)$ is regular) which would not be possible with Dzinotyiweyi's definitions [**11**].

The material falls into three chapters, each chapter consists of some sections and starts with a very short introduction, describing the material included. We have tried to make the text self-contained, up to certain extent.

Chapter one is devoted to a moderate discussion on preliminaries, according to our requirments.

Chapter two which is based on our work in [**24**] is devoted to introducing weighted semigroups (S,w), and studying some famous function spaces on them, especially the relations between $C_0(S,w)$ and other function spaces are investigated. In fact this chapter is a complement to [**32**]. One of the main features of this chapter is that the background set is a semigroup and not a group. In some instances we have even tried to drop some conditions (such as local compactness, or being Hausdorff) from the semigroup. Moreover, up to the best of our knowledge, for the first time, we have investigated, the connection between the translation invariance of $C_0(S)$ and $C_0(S,w)$, and consequently the connection between the topological structure of S, and the translation invariance of $C_0(S,w)$ has been investigated. This will be useful in investigating the relation between $C_0(S,w)$ and other function spaces.

Chapter three, is devoted to introducing means, homomorphisms and compactifications, and studying the relations between m-admissible subalgebra of $C(S,w)$ and compactifications of (S,w). This chapter shows that the one to one correspondence between m-admissible subalgebras, and compactifications reduces to the inclusion relation, in the case of weighted semigroups, i.e. compactifications lose a great deal of their importance in this case. Moreover we show that the existence of compactifications is independent of the definition of a mean (for which there has been quite a few different ones). This, in fact, is a consequence of the definition of a homomorphism between two weighted semigroups.

For the analytic background of this thesis, we refer to **[6]**,[7],**[20]**, while for topological background we refer to [**23**].

Finally, I take this occasion to record my debt to many people who have assisted me during the period of my research. First and foremost, my supervisor Professor M.A. Pourabdollah who deserves my profound gratitude for his enormous help, valuable guidance, and repetitious encouragement. I would also like to thank the best friend Dr. H.R. Ebrahimi-Vishki for his valuable suggestion, and also Drs A. Rejali, and M. Lashkarizadeh-Bami for providing me with some useful references.

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CHAPTER ONE

PRELIMINARIES

This chapter, which falls into four sections, is a moderate survey of the algebraic and topological theory of semigroups, emphasizing those aspects of the theory that will be used in the sequel. For notation and terminology we shall follow Berglund et al. [**6**], as far as possible.

1. Algebraic and Topological Semigroups

(1.1) Definitions. By a *semigroup* we shall mean a non-empty set S on which an associative binary operation, which is usually referred to as the *multiplication* of S, written multiplicatively, is defind. Hereafter S will be at least a semigroup. Every non-empty subset of S which is a semigroup under the restriction of the multiplication of S, is called a *subsemigroup* of S.

For $s, t \in S$ we write $\rho_t(s) = st = \lambda_s(t)$. For subsets A, B of S we write $As = \rho_s(A)$ and $sA = \lambda_s(A)$, and $AB = \{st : s \in A, t \in B\}$.

An element $1 \in S$ is said to be a *left* (resp. *right*) *identity* of S if $1s = s$ (resp. $s1 = s$), for all $s \in S$. A left identity that is also a right identity is called an *identity*.

An element $z \in S$ is called a *left* (resp. *right*) *zero* of S if $zs = z$ (resp. $sz = z$), for all $s \in S$. (If z is both left and right zero, it is called a zero.) If all elements of S are left (resp. right) zeroes, then S is called a *left* (resp. *right*) *zero semigroup*.

(1.2) Examples. (i) The set $S = N$ under the multiplication $m \odot n = max\{m, n\}$, $(m, n \in N)$ is a semigroup with identity (i.e. 1).

(ii) The set $S = R$ under the multiplication $s \odot t = s$ is a left zero semigroup.

(1.3) Definitions. Let S be a semigroup with a topology. Then S is a *right* (resp. *left*) *topological semigroup* if for each $s \in S$ the mappings ρ_s (resp. λ_s) is continuous. For a right topological semigroup S, $\Lambda(S)$ defined as, $\Lambda(S) = \{s \in S : \lambda_s \text{ is continuous }\}$, is called the *topological center* of S; note that it may be empty, [**6**; page 29]. If S is both right and left topological, it is called a *semitopological semigroup*; this is equivalent to the separate continuity of the multiplication mapping $(s, t) \mapsto st : S \times S \longrightarrow S$. If the latter is (jointly) continuous, S is said to be a *topological semigroup*. Right, left and semitopological groups can be defined analogously by exchanging semigroup with a group. A group which is a topological semigroup is called a *topological* *group* if the inversion mapping, i.e. $s \mapsto s^{-1} : S \longrightarrow S$, is continuous.

A mapping θ from S into another semigroup T is called a *homomorphism* if $\theta(s_1 s_2) = \theta(s_1)\theta(s_2)$ for all $s_1, s_2 \in S$. A mapping θ from a right topological semigroup S onto a right topological semigroup T, which is both an isomorphism, and a homeomorphism is called a *topological isomorphism* . If such a mapping exists, then S and T are said to be *topologically isomorphic.*

(1.4) Examples. (i) With respect to the usual topology, $\mathbf{Q}, \mathbf{R}$ and $\mathbf{C}$ are topological groups, under addition, and topological semigroups under multiplication. Under multiplication the circle group $\mathbf{T}$ ($:= \{z \in \mathbf{C} : \mid z \mid = 1\}$) is a compact topological group, and the unit disk $\mathbf{D}$ ($:= \{z \in \mathbf{C} : \mid z \mid \leq 1\}$) is a compact topological semigroup.

(ii) Every right zero (or left zero) semigroup is a topological semigroup in any topology.

(iii) Let S be a locally compact, noncompact, Hausdorff, right topological semigroup and let $S_\infty = S \cup \{\infty\}$ be the one-point compactification of S. S_∞ has a natural semigroup structure, namely one that makes S a subsemigroup and ∞ a zero. It is easy to check that, S_∞ is a right topological semigroup if and only if for each $s \in S$, ρ_s is a proper map; that is, for each compact subset K of S, ${\rho_s}^{-1}(K)$ is compact in S. For example, this is trivially satisfied when S is a group; and it fails to hold when S has a right zero, [**6**; 1.3.3(d)].

2. Means and Introversion of Function Spaces

(2.1) Definitios. Let Y be a non-empty set, we denote by $B(Y)$ the C^*-algebra of all bounded complex-valued functions on Y; if Y is a topological space, the C^*- subalgebra of all continuous elements of $B(Y)$ is denoted by $C(Y)$. If Y is locally compact, then $C_0(Y)$ denotes the C^*-subalgebra of $C(Y)$ consisting of the functions that vanish at infinity. (The elements of $C_0(Y)$ which have compact support is denoted by $C_{00}(Y)$.) For a topological vector space Y, the space of all affine elements of $C(Y)$ will be denoted by $AF(Y)$.

For $f \in B(Y)$ the functions Ref, Imf and $\overline{f}$ are defined as

$$(Ref)(y) = Re(f(y)) \quad , \quad (Imf)(y) = Im(f(y)) \quad , \quad \overline{f}(y) = \overline{f(y)}.$$

And $\| f \|$ is the uniform (or supremum) norm given by $\| f \| = sup\{| f(y) |: y \in Y\}$.

(ii) If X and Y are Banach spaces or, more generally, locally convex topological vector spaces, then $L(X, Y)$ denotes the vector space of all continuous linear mappings from X into Y. $L(X, X)$ is denoted by $L(X)$, and $L(X, C)$, the dual space of X, is denoted by X^*. If $A \subseteq X$ and $B \subseteq X^*$, then $\sigma(A, B)$

denotes the weakest topology on A relative to which the restriction to A of each member of B is continuous. A net $\{x_\alpha\}$ in A, $\sigma(A,B)$-converges to $x \in A$ if and only if $z^*(x_\alpha) \to z^*(x)$ for all $z^* \in B$. With the topology $\sigma(X,B)$, X is a locally convex topological vector space. $\sigma(X,X^*)$ is called the weak topology of X. Dually, $\sigma(B,A)$ is the weakest topology on B relative to which the mapping $z^* \to z^*(x) : B \to C$ is continuous for each $x \in A$. $\sigma(X^*,X)$ is called the weak* topology of X^*. If X and Y are Banach space, then $L(X,Y)$ is a Banach space under the uniform operator norm

$$\| U \| = sup\{\| Ux \| \ : \ \| x \| = 1\} \qquad (U \in L(X,Y)).$$

We denote the closure of a set A in a topological space by $\overline{A}$. If $\overline{A}$ is a subset of a locally convex topological vector space, then coA denotes the the convex hull of A.

(iii) Let $\mathcal{F}$ be a conjugate closed linear subspace of $B(Y)$ containing the constant functions; a member μ of $\mathcal{F}^*$ is said to be a *mean* on $\mathcal{F}$ if $\mu(1) = 1 = \|\mu\|$, where $\|.\|$ indicates the norm of $\mathcal{F}^*$. The set of all means on $\mathcal{F}$ is denoted by $M(\mathcal{F})$. If $\mathcal{F}$ is an algebra, then $\mu \in M(\mathcal{F})$ is called a *multiplicative mean* if $\mu(fg) = \mu(f)\mu(g)$ for all $f, g \in \mathcal{F}$. The set of all such means on $\mathcal{F}$ will be denoted by $MM(\mathcal{F})$. (For example, let $S = [0,1]$, with the ordinary multiplication and usual topology, then μ which is defined on

$C(S)$ by

$$\mu(f) = \int_0^1 f(x)\,dx$$

is a mean on $C(S)$).

Hereafter, $\mathcal{F}$ will be at least a conjugate closed subalgebra (resp. linear subspace) of $B(Y)$ containing the constant functions, whenever we deal with $MM(\mathcal{F})$ (resp. $M(\mathcal{F})$), and any mention of topology on $MM(\mathcal{F})$ and $M(\mathcal{F})$ refers to the relative weak* topology. We also define the *evaluation mapping* $\varepsilon : Y \longrightarrow MM(\mathcal{F})$ (resp. $: Y \longrightarrow M(\mathcal{F})$), by $\varepsilon(s)(f) = f(s) \quad (f \in \mathcal{F})$. Under these assumptions we have the following result.

(2.2) Theorem. *$MM(\mathcal{F})$ and $M(\mathcal{F})$ are* weak* *compact. Also $\varepsilon(S)$ and $co(\varepsilon(S))$, are* weak* *dense in $MM(\mathcal{F})$ and $M(\mathcal{F})$, respectively. Furthermore, $\mathcal{F}^*$ is the* weak* *closed linear span of $\varepsilon(S)$*; [**6**; 2.1.8].

(2.3) Remark. The mapping $\hat{} : \mathcal{F} \longrightarrow C(MM(\mathcal{F}))$ (resp. $: \mathcal{F} \longrightarrow C(M(\mathcal{F}))$), defined by $\hat{f}(\mu) = \mu(f) \quad (f \in \mathcal{F},\ \mu \in MM(\mathcal{F})$ (resp. $\mu \in \mathcal{M}(\mathcal{F})$)), is a multiplicatively linear (resp. linear) isometry that preserves complex conjugation. Moreover,

$\varepsilon^*(\hat{f}) = f$, where $\varepsilon^* : C(MM(\mathcal{F})) \longrightarrow \mathcal{B}(\mathcal{Y})$ (resp. $: C(M(\mathcal{F})) \longrightarrow \mathcal{B}(\mathcal{Y})$) denotes the dual mapping of ε. The next result, whose first version is a special case of the Gelfand-Naimark reperesentation theorem for commutative C^*-algebras, [**35**; 11.18], establishes suitable conditions on $\mathcal{F}$, under which

$\hat{}: \mathcal{F} \longrightarrow C(MM(\mathcal{F}))$ is onto, and $\hat{}: \mathcal{F} \longrightarrow C(M(\mathcal{F}))$ is onto $AF(M(\mathcal{F}))$.

(2.4) Teorem. *Every C^*- subalgebra (resp. conjugate closed Banach subspace) $\mathcal{F}$ of $B(Y)$ containing the constant functions, is isometrically isomorphic to $C(MM(\mathcal{F}))$ (resp. $AF(M(\mathcal{F}))$), under the algebra isomorphism (resp. linear isomorphism) $\hat{}$ with the inverse ε^**; [**6**; 2.1.9 & 2.1.10].

Now we will give some information about a typical subspace $\mathcal{F}$ of $B(S)$, where S is a semigroup.

(2.5) Definitions. (i) For every $s \in S$, *left* and *right translation oprators* L_s and R_s on $B(S)$ are defined by $L_sf(t) = f(st)$ and $R_sf(t) = f(ts)$, $(t \in S,\ f \in B(S))$. $\mathcal{F}$ is called *left* (resp. *right*) *translation invariant* if for all $s \in S$, $L_s\mathcal{F} \subseteq \mathcal{F}$ (resp. $R_s\mathcal{F} \subseteq \mathcal{F}$). If $\mathcal{F}$ is both left and right translation invariant, it is called *translation invariant.*

(ii) Let $\mathcal{F}$ be translation invariant. For $\mu \in \mathcal{F}^*$, the *left* and *right introversion oprators* T_μ and U_μ, from $\mathcal{F}$ into $B(S)$ are defined by $(T_\mu f)(s) = \mu(L_sf)$ and $(U_\mu f)(s) = \mu(R_sf)$, $(s \in S,\ f \in \mathcal{F})$.

A translation invariant subspace $\mathcal{F}$ is said to be *left* (resp. *right*) *introverted* if $T_\mu\mathcal{F} \subseteq \mathcal{F}$ (resp. $U_\mu\mathcal{F} \subseteq \mathcal{F}$) for all $\mu \in M(\mathcal{F})$ (or equivalently, for all $\mu \in \mathcal{F}^*$). If $\mathcal{F}$ is also an algebra, and the inclusions hold for $\mu \in MM(\mathcal{F})$ only, then $\mathcal{F}$ is called *left* (resp. *right*) *m-introverted.* $\mathcal{F}$ is called *introverted* (resp. *m-introverted*), if it is both left and right introverted (resp.

m-introverted).

(iii) An *admissible subspace* of $B(S)$ is a conjugate closed, translation invariant, left introverted Banach subspace of $B(S)$ containing the constant functions. An *m-admissible subalgebra* of $B(S)$ is a translation invariant, left m-introverted C^*-subalgebra of $B(S)$ containing the constant functions.

(iv) Let $\mathcal{F}$ be either left m-introverted and $\mu, \nu \in MM(\mathcal{F})$, or left introverted and $\mu, \nu \in \mathcal{F}^*$. Then we define the product $\mu\nu$ by $\mu \circ T_\nu$. Using the right m-introversion or right introversion of $\mathcal{F}$, we have also another product $\mu * \nu$, which is defined by $\nu \circ U_\mu$. These are called the *Arens products*, (which, in functional analysis are usually constructed in a different way that makes its existence clear but is less immediately accessible). An introverted subspace $\mathcal{F}$ of $B(S)$ is called *Arens regular* if $\mu\nu = \mu * \nu$ for all $\mu, \nu \in \mathcal{F}^*$.

Trivially, with the weak* topology the mappings $(\mu, \nu) \longrightarrow \mu\nu$ and $(\mu, \nu) \longrightarrow \mu * \nu$ are right and left continuous, respectively. These Arens products may induce a desirable semigroup structure on each of $MM(\mathcal{F})$, $M(\mathcal{F})$ and $\mathcal{F}^*$, as the next result demonstrates for the product $\mu\nu$.

(2.6) Theorem. (i) Let $\mathcal{F}$ be a translation invariant subalgebra (resp. subspace) of $B(S)$. Then for each $f \in \mathcal{F}$, $\{T_\mu f : \mu \in MM(\mathcal{F})\}$ (resp. $\{T_\mu f : \mu \in M(\mathcal{F})\}$) is the pointwise closure in $B(S)$ of $R_S f$ (resp. $co(R_S f)$).

(ii) Let S be a semigroup, and let $\mathcal{F}$ be an m-admissible subalgebra (resp.

admissible subspace) of $B(S)$. Then $MM(\mathcal{F})$ (resp. $M(\mathcal{F})$) supplied with the weak* topology and multiplication $\mu\nu$ is a compact right topological semigroup (resp. affine semigroup), under which $\varepsilon : S \longrightarrow MM(\mathcal{F})$ (resp. $:S \longrightarrow M(\mathcal{F})$) is a homomorphism with $\varepsilon(S) \subseteq \Lambda(MM(\mathcal{F}))$ (resp. $co(\varepsilon(S)) \subseteq \Lambda(M(\mathcal{F}))$; [**6**; 2.2.11].

Of course, if $\mathcal{F}$ is an admissible subspace, then $\mathcal{F}^*$, under $\mu\nu$ and the weak* topology is also a right topological affine semigroup.

For example if S is a compact semitopological semigroup, then $C(S)^*$ is introverted, and hence both $\mu\nu$ and $\mu * \nu$ are defined on $C(S)^*$ and

(a) $\mu\nu = \mu * \nu$ for all $\mu, \nu \in C(S)^*$

(b) with respect to the weak* topology and multiplication $(\mu, \nu) \longrightarrow \mu\nu$, $C(S)^*$ is a semitopological semigroup and hence $M := M(C(S))$ is a compact semitopological affine semigroup, and

(c) if S is a topological semigroup, then so is M; [**6**; 2.2.12].

3. Semigroup Compactifications

In what follows, unless otherwise stipulated, S will be at least a semitopological semigroup.

(3.1) Definitions. (i) By a *semigroup compactification* (or simply, a *compactification*) of S we mean a pair (ψ, X), where X is a compact, Hausdorff,