بسم الله الرحمن الرحيم

JORDAN C-DYNAMICAL SYSTEMS*

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پیوست ....

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Introduction:

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics. Let E be a complex Hilbert space, regarded as the "state space" of a quantum mechanical system. Let $\mathcal{H}(E)$ be the real vector space of all bounded self-adjoint linear operators on E, interpreted as the (bounded) observables of the system. In 1932, P. Jordan observed that $\mathcal{H}(E)$ is an (nonassociative) algebra via the anticommutator product

$$x \circ y := (xy + yx)/2.$$

Unlike the (associative) operator product xy, the anticommutator product preserves observables and has therefore a physical meaning.

In recent years Jordan algebras have found interesting applications to other areas of mathematics, e.g., to complex analysis in finite and infinite dimensions, to harmonic analysis on homogeneous spaces, to operator theory, and to the foundation of quantum mechanics.

C^*-algebras are more widely known and, as we shall see, JB algebras are so close to C^*-algebras that the phenomena we want to describe algebrically are often well enough described in terms of C^*-algebras.

We have devided the thesis into four chapters.

In the first chapter we give the necessary background of structure of commutators of operators and show what the commutator of two operators on a separable Hilbert space looks like.

In the second chapter we shall study basic property of JB and JB^*-algebras, JC and JC^*-algebras. The purpose of this chapter is to describe derivations of reversible JC-algebras in term of derivations of $B(H)$ which are well understood.

In chapter three we shall show that the tensor product of two Jordan generators is an infinitesimal generator on their special tensor products.

Finally, the object of fourth chapter is to develop a cohomology for Jordan operator algebras. We introduce two notions of amenability of JC^*-algebras and study module derivation of JB^*-algebras and tensor products of JC^*-algebras and nuclear JC^*-

algebras. In particular we prove the continuity of module derivation for certain JB^*-algebras and we will show that a JC^*-algebra is amenable if and only if it is nuclear. Finally we state a generalization of the Haagerup's theorem for certain non-associative algebras.

Contents:

Chapter IV. Amenability of Jordan Operator Algebras

4.1. Amenable Banach algebras

4.2. Module derivations of JB^*-algebras

4.3. Tensor product and nuclear JC^*-algebras

Chapter I
Commutators of Operators

In this chapter we give the necessary background of structure of commutators of operators and what the commutator of two operators on a separable Hilbert space can look like.

1.1. C^*-algebras

We begin by defining a number of concepts that make sense in any algebra with an involution.

An involution on an algebra A is a conjugate-linear map $a \longrightarrow a^*$ on A, such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. The pair $(A, *)$ is called an involutive algebra, or a $*$-algebra. If S is a subset of A, we set $S^* = \{a^* \mid a \in S\}$, and if $S^* = S$ we say S is self-adjoint. A self-adjoint subalgebra B of A is a $*$-subalgebra of A.

A Banach $*$-algebra is a $*$-algebra A together with a complete submultiplicative norm such that $||a^*|| = ||a||$ $(a \in A)$. If, in addition, A has a unit such that $||1|| = 1$, we call A a unital Banach $*$-algebra.

A C^*-algebra is a Banach $*$-algebra such that

$$||a^*a|| = ||a||^2 \quad (a \in A).$$

A representation of a C^*-algebra A is a pair (H, φ) where H is a Hilbert space and $\varphi : A \longrightarrow B(H)$ is a $*$-homomorphism. We say (H, φ) is faithful if φ is injective.

If $(H_\lambda, \phi_\lambda)_{\lambda \in \Lambda})$ is a family of representations of A, their direct sum is the representation (H, ϕ) got by setting $H = \oplus_\lambda H_\lambda$, and $\phi(a)((x_\lambda)_\lambda) = (\phi_\lambda(a)(x_\lambda))_\lambda$ for all $a \in A$ and all $(x_\lambda)_\lambda \in H$. It is readily veryfied that (H, ϕ) is indeed a representation of A. If for each non-zero element $a \in A$ there is an index λ such that $\phi_\lambda(a) \neq 0$, then (H, ϕ) is faithful.

Recall now that if H is an inner product space (that is, a pre-Hilbert space), then there is a unique inner product on the Banach space completion $\hat{H}$ of H extending the inner product of H and having as its associated norm the norm of $\hat{H}$. We call $\hat{H}$ endowed with this inner product the Hilbert space completion of H.

With each positive linear functional, there is associated a representation. Suppose that τ is a positive linear functional on a C^*-algebra A.

Setting $$N_\tau = \{a \in A \mid \tau(a^*a) = 0\},$$

it is easy to check that N_τ is closed left ideal of A and that the map

$$(A/N_\tau)^2 \longrightarrow C, (a + N_\tau, b + N_\tau) \longrightarrow \tau(b^* a),$$

is a well-defined inner product on A/N_τ. We denote by H_τ the Hilbert completion of A/N_τ.

If $a \in A$, define an oprator $\phi(a) \in B(A/N_\tau)$ by setting

$$\phi_\tau : A \longrightarrow B(H_\tau), a \longrightarrow \phi_\tau(a),$$

is a $*$-homomorphism.

The representation (H_τ, ϕ_τ) of A is the Gelfand-Naimark-Segal representation associated to τ.

If A is non-zero, we define its universal representation to be the direct sum of all the representetion (H_τ, ϕ_τ), where τ ranges over S(A).

1.1.1. Theorem (Gelfand-Naimark) [29]

If A is a C^*-algebra , then it has a faithful representation specifically, its universal representation is faithful.

1.2. Commutators of operators

Which operator C is commutator; i.e., when can C be expressed in the form $C = [A, B] = AB - BA$, where A and B are bounded linear operators on the same Hilbert space H? If H is finite dimensional, the answer is classical and easy stated: a necessary and sufficient condition that C be a commutator is that trace $C = 0$.

In more interesting case of a separable, infinite dimensional Hilbert space, the solution of the problem is quite a different form.

1.2.1. Theorem [24]

Suppose A and T are elements of a Banach algebra and suppose Q is the commutator of A and T,

$$Q = AT - TA.$$

Further, if Q commutes with A,

$$QA = AQ,$$

then Q is quasi-nilpotent, that is,

$$\lim_{n\to\infty} ||Q^n||^{1/n} = 0.$$

It will be recalled that the original discovery in this connection was made by Winter, who showed [47] that the identity operator is not a commutator. This negative result, which of course implies that no non-zero scalar is a commutator, was later obtained in a different way by Wielandt, who pointed out [46] that this much at least remains valid in an arbitrary normed ring with unit.

Subsequently, it was observed by Halmos [16] that the theorem of Winter-Wielandt, when applied to the residue class ring of the bounded operators module the ideal of compact operators, leads to the conclusion that no operator of the form $\lambda I + C$ where , $\lambda \neq 0$ and C is compact, is a commutator.

Let H be a separable Hilbert space. The algebra of all bounded linear operators on H is denoted by $B(H)$, and it is convenient to introduce at this time notation for some pertinent subset of $B(H)$: the ideal of compact operators will be denoted by (J), the class of all operators that are congruent mod(J) to non-zero scalars will be denoted by (S), and the complement in $B(H)$ of the disjoint union $(J) \cup (S)$ will be denoted by (F). The set (S) thus consists entirely of noncommutators, that every compact operator is a commutator