IN THE NAME OF GOD

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Superpreholomorphic sections on a complex supermanifold

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Dedicated to

A person

who gave new life pure Islam

at the time of peresent ignorance

and his two sons

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Abstract

In this paper we construct the B_L-superbialgebra C_L and prove that $(C_L^{\otimes r})^*$ is isomorphic to B_L-superalgebra of power series of r variables with coefficients in F. Hence it contains all of germs of superholomorphic functions at each point of a complex super manifold of dimension (m, n), where $n \leq L$, $r = 2^{L-1}(m+n)$. Then we introduce the notion of superpreholomorphic sections on the complex supermanifold. In more details we compute two cohomology groups of the sheaf of superpreholomorphic sections on a complex supermanifold. Finally we discuss the supercoderivations of C_L.

Contents

Chapter 0 Introduction 1

Chapter 1 Preliminaries 3

1.1 Hopf algebra 4

1.2 Coderivations of a coalgebra 9

1.3 The bialgebra B(V) 11

1.4 sheaf and cohomology 14

1.5 superalgebra 17

1.6 Super Hopf algebra 22

1.7 Supermaninifold 30

Chapter 2 Superbialgebra C_L and $(C_L^{\otimes r})^*$ 34

2.1 Superbialgebra C_L 35

2.2 Superbialgebra $(C_L^{\otimes r})^*$ 40

Chapter 3 Superpreholomorphic sections on complex supermanifold 46

3.1 Superpreholomorphic sections on complex supermanifold 47

3.2 Sheaf of germs of Superpreholomorphic sections and two cohomology classes of it 49

Chapter 4 On the Supercoderivations of $SH(B_L^{m,n})$ 54

4.1 B_L-super Hoph algebra SH 55

4.2 Supercoderivations of an A-supercoalgebra 58

References 67

Chapter 0

Introduction

The purpose of noncommutative geometry is to extend the correspondence between geometric spaces and commutative algebras to the noncommutative case in the framework of complex analysis. Such spaces arise both in mathematics and in quantum physics. The term "quantum groups" was popularized by Drinfeld [4]. It stands for certain special Hopf algebras which are nontrivial deformations of the enveloing Hopf algebras of semisimple Lie algebras of the algebras of regular functions on the corresponding algebraic groups.

The preholomorphic section is defined by Nekooie [8] and then is developed on complex manifolds in [5].Sabsevary [10] defined superpreholomorphic sections on $\mathbb{C}^{1,1}$ and also the supercoderivations of a supercoalgebra.

We assume the reader is familiar with the definitions of Hopf algebra (see [1] & [11]), sheaf and cohomology group (see [12]), superalgebra and super manifold (see [2],[3],[6] and [9]).

This thesis is organized as follows.In the first chapter we have the preliminaries.Then in the secound chapter we construct a B_L-superbialgebra whose dual is isomorphic to the superalgebra of power series of r-variables with coefficients in F. In the next chapter we introduce the notion of superpreholomorphic sections and prove that the set of germs of superpreholomorphic sections is a sheaf and then we compute two cohomology groups of this sheaf. In the fourth chapter we discuss the supercoderivations of an A-supercoalgebra and we find the general form of the supercoderivations of the C_L.

Chapter 1

Preliminaries

1.1 Hopf algebra

We assume that $k = \mathbb{R}, \mathbb{C}$ or any field.

Definition 1.1.1 A k-**algebra** is a triple (A, M, U) with A, a k-module and two k-linear maps $M : A \otimes_k A \longrightarrow A$ (called multiplication) and $U : A \longrightarrow k$ (called unit map) such that the following diagrams commute.

$$\begin{array}{ccc} A \otimes_k A \otimes_k A & \xrightarrow{I \otimes M} & A \otimes_k A \\ M \otimes I \downarrow & & \downarrow M \\ A \otimes A & \xrightarrow{M} & A \end{array} \qquad \text{(associativity)}$$

and

$$\begin{array}{ccccc} & & A & & \\ & \swarrow & & \searrow & \\ k \otimes A & & \uparrow M & & A \otimes k \\ U \otimes I \nwarrow & & & & \nearrow I \otimes U \\ & & A \otimes_k A & & \end{array} \qquad \text{(unitary property)}$$

Definition 1.1.2 A k-**coalgebra** is a triple (C, Δ, ϵ) with C a k-module and $\Delta : C \longrightarrow C \otimes_k C$ and $\epsilon : C \longrightarrow k$ two k-linear maps such that the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes_k C \\ \Delta \downarrow & & \downarrow \Delta \otimes I \\ C \otimes_k C & \xrightarrow{I \otimes \Delta} & C \otimes_k C \otimes_k C \end{array} \qquad \text{(coassociativity)}$$

and

$$\begin{array}{ccccc} & & C & & \\ & \swarrow & & \searrow & \\ k \otimes C & & \downarrow \Delta & & C \otimes k \\ & \epsilon \otimes I \nwarrow & & \nearrow I \otimes \epsilon & \\ & & C \otimes_k C & & \end{array} \qquad \text{(counitary property)}$$

If we denote $\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$, then we have

$$(\Delta \otimes I) \circ \Delta(c) = (I \otimes \Delta) \circ \Delta(c),$$

$$\sum_{(c)} \Delta(c_{(1)}) \otimes c_{(2)} = \sum_{(c)} c(1) \otimes \Delta(c_{(2)}) = \sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes c_{(3)},$$

and $(I \otimes \epsilon) \circ \Delta(c) = (\epsilon \otimes I) \circ \Delta(c) = \text{id}$,

$\sum_{(c)} \epsilon(c_{(1)}) \otimes c_{(2)} = \sum_{(c)} c_{(1)} \otimes \epsilon(c_{(2)}) = c$. Let $\tau : C \otimes_k C \longrightarrow C \otimes_k C$ be twist map; the bilinear map defined by $\tau(x \otimes y) = y \otimes x$. Then a k-coalgebra (C, Δ, ϵ) satisfying $\tau \circ \Delta = \Delta$ is said to be **cocommutative**.

Definition 1.1.3 Let $(C, \Delta_C, \epsilon_C), (D, \Delta_D, \epsilon_D)$ be two k-coalgebras, then a k-linear map $f : C \longrightarrow D$ is a **k-coalgebra morphism** if the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \Delta_C \downarrow & & \downarrow \Delta_D \\ C \otimes_k C & \xrightarrow{f \otimes f} & D \otimes_k D \end{array}$$

So $\sum_{(c)} f(c_{(1)}) \otimes f(c_{(2)}) = \sum_{(f(c))} f(c)_{(1)} \otimes f(c)_{(2)}$,

and $(\epsilon_D \circ f)(c) = \epsilon_C(c)$ for all $c \in C$.

Lemma 1.1.4 Let $(C, \Delta_C, \epsilon_C)$ be a k-coalgebra,

a) Let D be a subspace of C, satisfying the condition $\Delta_C(D) \subseteq D \otimes_k D$, then $(D, \Delta|_D, \epsilon|_D)$ becomes a k-coalgebra, called **k-subcoalgebra** of C and $i : D \longrightarrow C$, the natural injection map is a k-coalgebra morphism.

b) Let $(D, \Delta_D, \epsilon_D)$ be another k-coalgebra then $(C \otimes_k D, \Delta_{C\otimes D}, \epsilon_{C\otimes D})$ is a k-coalgebra where $\Delta_{C\otimes D} = (I \otimes \tau \otimes I) \circ (\Delta_C \otimes \Delta_D)$ and $\epsilon_{C\otimes D} = \epsilon_C \otimes \epsilon_D$.

If we have $\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$, $\Delta(d) = \sum_{(d)} d_{(1)} \otimes d_{(2)}$, then

$\Delta_{C\otimes D}(c \otimes d) = \sum_{(c)} \sum_{(d)} c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)}$, and

$\epsilon_{(C\otimes D)}(c \otimes d) = \epsilon_C(c)\ \epsilon_D(d)$, for all $c \in C$ and $d \in D$.

Proof. Straightforward.

Lemma 1.1.5 Let V, W be two k-linear spaces and $V^* = \mathrm{Hom}_k(V, k)$. If we define $\rho : V^* \otimes W^* \longrightarrow (V \otimes W)^*$ by $\rho(\alpha \otimes \beta)(x \otimes y) = \langle \alpha, x\rangle\langle \beta, y\rangle$, where $\alpha \in V^*, \beta \in W^*, x \in V$ and $y \in W$ then ρ is an injective linear map.

Proof. [8. Appendix I].

Proposition 1.1.6 Let (C, Δ, ϵ) be a k-coalgebra. Setting

$M : C^* \otimes C^* \xrightarrow{\rho} (C \otimes C)^* \xrightarrow{\Delta^*} C^*$ and $U : k \xrightarrow{\sim} k^* \xrightarrow{\epsilon^*} C^*$, (C^*, M, U)

becomes a k-algebra which we call the **dual k-algebra of C**.

Proof. [8, 1.1.1].

Proposition 1.1.7 Let (A, M, U) be an algebra with finite dimensional k-linear space, then $\rho : A^* \otimes A^* \longrightarrow (A \otimes A)^*$ turns out to be a bijective. We define $\Delta = \rho^{-1} \circ M^*$, $\epsilon = U^*$ then (A^*, Δ, ϵ) is a coalgebra, called the **dual**

k-**coalgebra** of A.

Proof. [8, 1.1.2].

Proposition 1.1.8 a) Let $f : C \longrightarrow D$ be a linear map of coalgebras. Then $f^* : D^* \longrightarrow C^*$ is an algebra map.

b) Let $f : A \longrightarrow B$ be a map of dimensional algebras. Then $f^* : B^* \longrightarrow A^*$ is a coalgebra map.

Proof. [8, 1.4.1 and 1.4.2].

Example 1.1.9 Let B be a k-vector space with the basis $\{c_i\}_{i=0}^{\infty}$. We define $\Delta : B \longrightarrow B \otimes_k B$ by $\Delta(c_n) = \sum_{i=0}^{n} c_i \otimes c_{n-i}$ and $\epsilon : B \longrightarrow k$ by $\epsilon(c_n) = \delta_{n,0}$. Then (B, Δ, ϵ) is a k-coalgebra.

One can see (1,p. 53-55) as the another reference of coalgebra.

Proposition 1.1.10 Let (H, M, U) be a k-coalgebra and (H, Δ, ϵ) be a coalgebra. The following conditions are equivalent.

i) M and U are coalgebra maps,

ii) Δ and ϵ are algebra maps,

iii) $\Delta(gh) = \sum_{(gh)} g_{(1)}h_{(1)} \otimes g_{(2)}h_{(2)}$,
$\Delta(1) = 1 \otimes 1$, $\epsilon(gh) = \epsilon(g)\epsilon(h)$, $\epsilon(1) = 1$.

Proof. [8, 3.1.1] or [1, 2.1.1].

Definition 1.1.11 Any system satisfying the conditions of (1.1.10) is called a k-**bialgebra**.

Example 1.1.12 Let B be k-coalgebra as in (1.1.9).

If we define $M(c_i \otimes c_j) = \begin{pmatrix} i+j \\ i \end{pmatrix} c_{i+j}$ and $U(1) = c_0$, then $(B, M, U, \Delta, \epsilon)$ is a k-bialgebra.

If we set $X_i = (c_i)^*$, for $i = 0, 1, 2, \ldots$, then $\{X_i\}_{i=0}^{\infty}$ is a basis of B^*, the dual algebra of B. Since we have

$$\begin{aligned}
\langle X_i X_j, c_n \rangle &= \langle \Delta^* \circ \rho(X_i \otimes X_j), c_n \rangle \\
&= \langle \rho(X_i \otimes X_j), \Delta(c_n) \rangle \\
&= \sum_{i=0}^{n} \langle \rho(X_i \otimes X_j), c_k \otimes c_{n-k} \rangle \\
&= \sum_{i=0}^{n} \langle X_i, c_k \rangle \langle X_j, c_{n-k} \rangle \\
&= \sum_{i=0}^{n} \delta_{i,k}\ \delta_{j,n-k} = \delta_{i+j,n} \\
&= \langle X_{i+j}, c_n \rangle.
\end{aligned}$$

So the multiplication in B^* is as in $k[X]$.

Therefore if we define $\varphi : B^* \longrightarrow k[X]$, by $\varphi(X_i) = X^i$, then φ is an isomorphism of k-algebras.

If C is a k-coalgebra and A is a k-algebra, we give $\mathrm{Hom}_k(C, A)$ an algebra structure by $f * g = M_A \circ (f \otimes g) \circ \Delta_C$ or $(f * g)(c) = \sum_{(c)} f(c_{(1)})\ g(c_{(2)})$, for all $f, g \in \mathrm{Hom}_k(C, A)$. ($f * g$ is said to be **convolution** of f and g) and the unit $U \circ \epsilon(c) = \epsilon(c)U(1_k)$ for all $c \in C$.

Suppose H is a bialgebra with underlying algebra H^A and coalgebra H^C. As the above way $\mathrm{Hom}(H^C, H^A)$ becomes an algebra and $U \circ \epsilon$ is its unit.

Definition 1.1.13 An element $S \in \mathrm{Hom}(H^C, H^A)$ which is inverse under $*$ is called an **antipode** for H.

A bialgebra with an antipode is a **Hopf algebra**.

Thus the k-linear map $S : H \longrightarrow H$ is an antipode if and only if

$S * I = I * S = U \circ \epsilon$ or $\sum_{(h)} h(1)S(h_{(2)}) = \sum_{(h)} S(h(1))(h_{(2)}) = \epsilon(h)U(1)$,

for all $h \in H$.

Of course if H has an antipode, it is unique. See the properties of the antipode S of a k-Hopf algebra H in [1, 2.1.4].

1.2 Coderivations of a coalgebra

Definition 1.2.1 Suppose (C, Δ, ϵ) is a coalgebr over k. A k-linear map $X : C \longrightarrow C$ is called a **coderivation** of C if the diagram

$$\begin{array}{ccc} C & \xrightarrow{X} & C \\ \Delta \downarrow & & \downarrow \Delta \\ C \otimes C & \xrightarrow{\overline{X}} & C \otimes C \end{array}$$

commutes where $\overline{X}(a \otimes b) = X(a) \otimes b + a \otimes X(b)$.

Proposition 1.2.2 Suppose (C, Δ, ϵ) is a k-coalgebra and $X : C \longrightarrow C$ is a coderivation of C. Then $X^* : C^* \longrightarrow C^*$ is a derivation of C^*.

Proof. By (1.1.6), (C^*, M, U) is a k-algebra with $M = \Delta^* \circ \rho$ and $U = \epsilon^*$.