IN THE NAME OF GOD

DYNAMICS AND INTEGRABILITY OF GENERALIZED SINE-GORDON EQUATIONS

By

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THESIS

SUBMITTED TO THE SCHOOL OF GRADUATE STUDIES IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY (Ph.D.)

IN

PHYSICS

SHIRAZ UNIVERSITY

SHIRAZ, IRAN

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October 1999

Dedicated

To

my wife *who bore hard during the course of my research*

ACKNOWLEDGMENT

I would like to express my sincere gratitude and appreciation to Prof. N. Riazi for his constant help, guidance and encouragement during the course of my research. Helpful discussions with Dr. M. H. Dehghani is gratefully acknowledged.

ABSTRACT

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We have studied a few different topics in the dynamics of the sine-Gordon equation and its modified forms.

We introduce a few generalizations of the sine-Gordon equation which lead to the ordinary sine-Gordon equation as limiting cases. We modify this in three ways. Firstly, we multiply the Lagrangian density of the sine-Gordon equation by a power-law function of the scalar field. The total energy for the static and dynamic solutions are obtained accordingly. We perform the calculation for the static case, and show that the total energy of the moving solitary waves satisfies the Einstein relation. In the second approach, we define the potential function as $V(\phi) = a(1 - \cos b\phi)$ in which a has different values for positive and negative ϕ and b is a constant. Finally, we modify the sine-Gordon system by adding the term $\epsilon(1 - \cos 2\phi)$ to the Lagrangian density of the ordinary sine-Gordon

system, and derive analytically its solitary wave solutions and some of their properties.

We study the dynamics of solitons in inhomogeneous media in two ways. The inhomogeneity of the first kind is introduced via spatially varying parameters a and b. Like a classical particle, the kink bypasses a potential barrier, as long as its kinetic energy is more than the potential height. However, if the barrier is thin enough, a 'tunneling effect' happens. Secondly, we implement the inhomogeneity into the sine-Gordon equation by replacing a with a function of position and adding a new term to the equation. The motion of the kink's center of mass is shown to be governed by the Newton's second law. We study some illustrative examples by numerical calculations.

Finally, we conjecture a method to determine whether a nonlinear model is integrable or not. In this method, we employ a linear perturbation analysis to investigate the relationship with integrability. According to our conjecture, a system is integrable, if the corresponding perterbative eigen-value equation has at most one bound state with zero frequency, which results from a symmetry of the solution. We study two kinds of dynamical equations. First we consider equations which are Lorentz covariant, and then equations which have Galilean invariance.

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Chapter 1

SOLITARY WAVES AND SOLITONS

1.1 Introduction

Description and interpretation of phenomena in the universe by using linear physics are more or less approximate. There are many phenomena in nature that we cannot easily describe and interpret, because their dynamical equation are nonlinear. Therefore, many scientists believe that one of the best fundamental routes to understanding the nature is tendency toward the nonlinear physics.

Although it is more than one century from the birth of the nonlinear physics, but there was no considerable progress until recently. One of the main reasons for this delay was computational difficulties. Furthermore, earlier in this century, the birth of quantum mechanics and its successful results attracted more physicists. Therefore, the nonlinear science was almost forgotten. Fortunately, during the past few decades scientists have paid much more attention to this subject, and it is observed that a considerable number of papers are allocated to the nonlinear science, particularly in fields such as physics, mathematics and chemistry. There are particular subjects such as chaos, fractals and solitons which cannot be interpreted by linear physics at all. These subjects and related nonlinear phenomena in physics, have created another subgroup in physics which is known as *nonlinear physics.*

This thesis is devoted to the soliton theory, which has proven to be a very attractive and exciting topic. This fascinating theory has been applied in various fields such as field theory, optics, fundamental particle physics and solid state. One of the most attractive characteristics of a soliton is its pseudo-particle behavior.

The main purpose of this thesis is to present this aspect of soliton behavior. The material is presented as follows. In this chapter, I briefly review the historical progress in the study of nonlinear systems, introduce a few fundamental definitions and give some examples. In the second Chapter, some applications of the sine-Gordon equation are presented. Because the main equation studied in this thesis is the sine-Gordon equation, I will discuss different types of soliton solutions of the sine-Gordon equation in Chapter three. By introducing a few generalizations of the sine-Gordon equation in Chapter four, two methods are examined which lead to the sine-Gordon equation as limiting cases. In Chapter five, I will deform the sine-Gordon equation in another way and derive its analytical and numerical solutions. In this case, the kink motion exactly resembles that of a classical particle moving in an external potential.

There are various methods for determining the integrability of nonlinear equations. These methods are usually cumbersome and tedious. In Chapter six, I will introduce a conjecture to determine the integrability of a class of nonlinear equations.

Finally, I will present the conclusions briefly and summarize the results in Chapter seven.

1.2 Solitary Waves and Solitons

Since solitary waves and soliton solutions are derived from nonlinear equations, it is preferable to define a nonlinear differential equation, [Boyce and Diprima, 1977]:

The differential equation

$$F(x, y, y', y'', ..., y^{(n)}) = 0 \tag{1.1}$$

is said to be *linear* if F is a linear function of the variables, $y, y', y'', ..., y^{(n)}$.