Shahid Bahonar University of Kerman
Faculty Of Mathematics and Computer Science
Department of Mathematics

# DEDEKIND MODULES AND DIMENSION OF MODULES

By
**S. Karimzadeh**

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Supervisor: ______________________
**Prof. R. Nekooei**

Readers: ______________________
Prof. E. Eslami

______________________
Prof. R. Ameri

______________________
Prof. S. Hedaiat

SHAHID BAHONAR UNIVERSITY OF KERMAN

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Author: **S. Karimzadeh**

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# Abstract

In this thesis, at first we obtain some equivalent conditions for a module over a Prüfer domain and derive properties of Dedekind modules over such a domain. We then obtain equivalent conditions for a finitely generated Dedekind module over an integrally closed ring and we characterize multiplication Dedekind modules. Finally, we prove the lying over and the going down theorems for modules and apply them to prove some results on the dimension of a module and its submodules.

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# Introduction

Dedekind domains have been studied for many years. Dedekind modules, were first defined by A.G.Naoum and F.H.Al-Alwan in 1996([5]). They generalized the notion of Dedekind domain to module theory. Later, M.Alkan, Y.Tiras in 2004 ([4]), M.Alkan, B.Sarac and Y.Tiras in 2005 ([3]) and B.Sarac, P. F.Smith and Y.Tiras in 2007([21]) generalized some other notions such as the notion of integrally closed modules to in order to characterize Dedekind modules and obtain some properties of Dedekind modules that are similar to the properties in ring theory. We aim to characterize Dedekind modules and submodules of Dedekind modules and $D_1$-modules.

Finally, we shall prove the lying over and going down theorems for modules, and then prove some results on the dimension of a module and its submodules.

# Chapter 1

# Preliminaries

Throughout this thesis, unless stated otherwise, all rings are commutative with identity and all modules are unitary. In this chapter, we first introduce the notion of a prime submodule. Secondly we introduce Dedekind modules and we shall review some researches into the topics related to Dedekind modules.

## 1.1 Prime Submodules

**Definition 1.1.1.** ([9]) An $R$-module $M \neq 0$ is called a *simple module* if it has no submodules except 0 and $M$.

Let $R$ be a ring and let $N$ be a submodule of an $R$-module $M$. The set

$$\{r \in R \mid rM \subseteq N\}$$

is denoted by $(N : M)$ and $(0 : M)$ is denoted by $Ann_R(M)$.

**Definition 1.1.2.** ([13]) Let $M$ be an $R$-module. A submodule $Q$ of $M$ is *primary* if for all $r \in R$ and $m \in M$, $rm \in Q$ and $m \notin Q$ imply that $r^n M \subseteq Q$ for some positive integer $n$.

If $Q$ is a primary submodule of $M$ then $\sqrt{(N : M}$ is a prime ideal of $R$.

**Definition 1.1.3.** ([14]) A submodule $N$ of an $R$-module $M$ is called *prime* if $N \neq M$ and given $r \in R$, $m \in M$, $rm \in N$ implies $m \in N$ or $r \in (N : M)$.

We note that when $R$ is a noncommutative ring, the statement "$rm \in N$" in the above definition is replaced by "$rRm \subseteq N$". Clearly, every prime submodule of a module is primary. Let $N$ be a submodule of $M$. It is easy to see that $N$ is a prime submodule of $M$ if and only if $p = (N : M)$ is a prime ideal of $R$ and the $\frac{R}{p}$-module $\frac{M}{N}$ is torsion-free. We shall call $N$ a $p$-prime submodule of $M$. Clearly, an ideal of a ring $R$ is prime if and only if it is a prime submodule of $R$ as $R$-module. In contrast with rings, modules may have no prime submodules. For example, when $p$ is a prime integer, the $\mathbf{Z}$-module $\mathbf{Z}(p^{\infty})$ has no prime submodule.

Let $M$ be an $R$-module. The set of all prime submodules and maximal submodules of $M$ are respectively denoted by $Spec(M)$ and $Max(M)$.

**Definition 1.1.4.** [15] The *radical submodule* $N$ of $R$-module $M$ is given by $rad_M(N) = \cap P$ where the intersection is over all prime submodules of $M$ containing $N$. If there is no prime submodule containing $N$, then we put $rad_M(N) = M$.

we denote the intersection of all prime submodules of an $R$-module $M$, by $rad_M(0)$ and the intersections of all proper maximal submodules by $Rad(M)$. The radicals of $R$ and an ideal $I$ of $R$ are denoted by $N(R)$ and $\sqrt{I}$ respectively.

A ring $R$ is called *Von Neumann regular* ring if for every $a \in R$ there exists an element $b \in R$ such that $a = aba$.

**Corollary 1.1.1.** *([14, Corollary page 62]) If $M$ is a module over a Von Neumann regular ring, then every primary submodule of $M$ is prime.*

**Proposition 1.1.2.** *([17, Proposition 1.2]) Let $R$ be any ring, $M$ and $M'$ left $R$-modules, and $\varphi \in Hom_R(M, M')$. Let $N$ be a prime submodule of $M'$ such that $\varphi(M) \nsubseteq N$. Then $\varphi^{-1}(N)$ is a prime submodule of $M$.*

**Definition 1.1.5.** Suppose that $M$ is an $R$-module and $P$ is a prime ideal of $R$. Put $S_P = R\backslash P$. Define the *distinguished submodule* $PM(S_P)$ as $\{x \in M : sx \in PM$, for some $s \in S\}$ of $M$.

**Note:** If $PM(S_P) \neq M$ then $PM(S_P)$ is a prime submodule of $M$.

**Example 1.1.3.** $(i)$ Let $M = \mathbf{Z} \oplus \mathbf{Z}$ be a $\mathbf{Z}$-module. If $P$ is a prime ideal of $\mathbf{Z}$ then $PM(S_P) = P \oplus P$ and $PM(S_P)$ is a prime submodule.
$(ii)$ Let $M = R$ be an $R$-module. If $P$ is a prime ideal of $R$ then $PM(S_P) = P$.

**Definition 1.1.6.** ([1]) The *dimension* of $M$ is the maximal positive integer $k$, such that there exist a chain of prime distinguished submodules of $M$ as follows:

$$N_0 \subset N_1 \subset \cdots \subset N_k$$

and we set $dimM = \infty$ if there is a chain of the above kind for every value of $n$.

**Definition 1.1.7.** ([7] An $R$-module $M$ is called a *multiplication* $R$-module, provided for every submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$.

For example every ring $R$ is a multiplication $R$-module.

**Proposition 1.1.4.** *([7, Proposition 3.4]) Let $M$ be a faithful multiplication $R$-module. Then $M$ is finitely generated if and only if $M \neq pM$ for all minimal prime ideals $p$ of $R$.*

In particular, Proposition 1.1.4 shows that if $R$ is a domain then any faithful multiplication module is finitely generated.

**Notation.** Let $A = (a_{ij}) \in M_{m\times n}(R)$ and $F$ be the free $R$-module $R^{(n)}$. We shall use the notation $\langle A\rangle$ for the submodule $N$ of $F$ generated by the rows of $A$, and the notation $(r_1, \ldots, r_m)A$, $r_i \in R$, for an element of $N$.

**Lemma 1.1.5.** *([10, Lemma 1.2]) Let $R$ be a principal ideal domain (PID) and $A \in M_n(R)$ , $det(A) \neq 0$ and $A' = (a'_{ij})$ be the adjoint matrix of $A$. Then $(x_1, ..., x_n) \in \langle A\rangle$, for some $x_i \in R$ $(1 \leq i \leq n)$ if and only if $det(A) \mid \sum_{i=1}^{n} x_i a'_{ij}$ , for every $j$, $1 \leq j \leq n$.*

**Definition 1.1.8.** ([10]) Let $R$ be a principal ideal domain (PID) and $J = \{j_1, ..., j_\alpha\}$ be a subset of $\{1, ..., n\}$ and let $p \in R$ be a prime element. A matrix $A \in M_n(R)$, $A = (a_{ij})$, is said to be a *p-prime matrix*(or simply prime) if $A$ satisfies the following conditions:

$(i)$ $A$ is upper triangular.

$(ii)$ For all $i$, $1 \leq i \leq n$, $a_{ii} = p$ if $i \in J$ and $a_{ii} = 1$if $i \notin J$.

$(iii)$ For all $i$, $1 \leq i \leq j \leq n$, $a_{ij} = 0$ except possibly when $i \notin J$ and $j \in J$.

Sometimes we call $J$ the set of integers associated with $A$ and denote it by $J_A$.

By $(i)$ and $(ii)$ it is clear that $det(A) = p^\alpha$.

**Example 1.1.6.** Let $R = \mathbf{Z}$ and $J = \{2, 4, 6\}$ and let $p \in \mathbf{Z}$ be a prime element.

$$A = \begin{pmatrix} 1 & a_{12} & 0 & a_{14} & 0 & a_{16} & 0 \\ 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a_{34} & 0 & a_{36} & 0 \\ 0 & 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & a_{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then matrix $A$ is prime matrix and $det(A) = p^3$.

**Theorem 1.1.7.** *([10, Theorem 2.5]) Every full rank prime submodule of $R^{(n)}$ is the row space of a prime matrix and vice versa.*

**Example 1.1.8.** ([10]) For every prime element $p \in \mathbf{Z}$, the prime submodules $N$ of $\mathbf{Z}^3 = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ such that $(N : \mathbf{Z}^3) = p\mathbf{Z}$ are as follows:

$$\langle \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle, \langle \begin{pmatrix} 1 & a_{12} & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle, \langle \begin{pmatrix} 1 & 0 & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & p \end{pmatrix} \rangle, \langle \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle,$$

$$\langle \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & a_{23} \\ 0 & 0 & p \end{pmatrix} \rangle, \langle \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \rangle, \langle \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \rangle,$$

where $0 \leq a_{ij} \leq p - 1$, $1 \leq i < j \leq 3$.

## 1.2 Dedekind Domains and Prüfer Domains

**Definition 1.2.1.** ([13]) Let $R$ be a ring. An element $a \in R$ is called a *zero-divisor* if there is an element $b \in R$, $b \neq 0$, such that $ab = 0$.

An element of $R$ which is not a zero-divisor is called *regular*.

**Definition 1.2.2.** ([13]) Let $R$ be a ring. A subset $S \subseteq R$, is called a *multiplicatively closed set* in $R$ if $0 \notin S$, $1 \in S$ and $ab \in S$ whenever $a \in S$ and $b \in S$.

**Example 1.2.1.** If $p$ is a prime ideal of $R$ then $S = R \backslash p$ is a multiplicatively closed set in $R$.

**Definition 1.2.3.** ([13]) Let $M$ be an $R$-module and $S$ be a multiplicatively closed set in $R$. The *localisation* of $M$ with respect to $S$ is constructed as follows.

Consider the set $M \times S$. Define a relation $\sim$ on $M \times S$ by condition, $(x,\ s) \sim (y,\ t)$ if and only if $s_1(tx - sy) = 0$, for some $s_1 \in S$. It is easy to verify that $\sim$ is an equivalence relation on $M \times S$. The set of equivalence classes is denoted by $S^{-1}M$. Any element of $S^{-1}M$ being an equivalence class containing $(x,\ s)$ is denoted by the symbol $(\frac{x}{s},\ x \in M,\ s \in S$. In particular for $M = R$, the set $S^{-1}R$ is defined.

If $p$ is a proper prime ideal of $R$ and $S = R \backslash p$ then we denote $S^{-1}R$ by $R_p$ and called it the quotient ring of $R$ with respect to $p$.

If $S$ is the set of all regular elements of $R$, then $S$ is a multiplicatively closed set in $R$. The ring $S^{-1}R$ is called the *total quotient* ring of $R$.

**Definition 1.2.4.** ([13]) A *valuation ring* is an integral domain $R$ with the property that if $I$ and $J$ are ideals of $R$ then either $I \subseteq J$ or $J \subseteq I$.

**Example 1.2.2.** ($i$) Any field $K$ is a Valuation ring.

($ii$) Let $K$ be a field and

$$R = \{\frac{f(x)}{g(x)} : f(x),\ g(x) \in K(x),\ deg\ f \leq deg\ g\}.$$

$R$ is a valuation ring.

**Definition 1.2.5.** ([13]) A *fractional ideal* of a ring $R$ is a subset $A$ of the total quotient ring $K$ of $R$ such that:

($i$) $A$ is an $R$-module;

($ii$) There is a regular element $d$ of $R$ such that $dA \subseteq R$.

For example every ideal of $R$ is a fractional ideal of $R$.

**Definition 1.2.6.** ([13]) A nonzero fractional ideal $A$ of a ring $R$ is said to be *invertible* if $A^{-1}A = R$, where $A^{-1} = \{x \in K : xA \subseteq R\}$.

**Definition 1.2.7.** ([13]) A domain $R$ is called a *Prüfer domain* if every nonzero finitely generated ideal of $R$ is invertible.

For example every principal ideal domain is a Prüfer domain.

The following theorem contains some basic properties of Prüfer domain.

**Theorem 1.2.3.** *([13, Theorem 6.6]) If $R$ is an integral domain, then the following statements are equivalent:*

($i$) *$R$ is a Prüfer domain;*

($ii$) *Every nonzero ideal of $R$ generated by two elements is invertible;*

($iii$) *$R_p$ is a valuation ring, for every prime ideal $p$ of $R$.*

**Definition 1.2.8.** ([13]) If $R$ is a ring and $K$ is its total quotient ring, then any ring $T$ such that $R \subseteq T \subseteq K$ is called an *overring* of $R$.

We give two characterizations of Prüfer domain in terms of their overrings.

**Theorem 1.2.4.** *([13, Theorem 6.10]) An integral domain $R$ is a Prüfer domain if and only if every overring of $R$ is a flat $R$-module.*

**Corollary 1.2.5.** *([13, Corollary 6.11]) If an integral domain $R$ is a Prüfer domain then every overring of $R$ is a Prüfer domain.*

**Theorem 1.2.6.** *([13, Theorem 6.13]) An integral domain $R$ is a Prüfer domain if and only if every overring of $R$ is integrally closed.*

**Theorem 1.2.7.** *([8, Theorem 2.7, page 155]) A finitely generated torsion-free module $M$ over a Prüfer domain $R$ is projective, and hence finitely presented. Moreover,*

$$M \cong I_1 \oplus ... \oplus I_n,$$

*where $I_1, ..., I_n$ are finitely generated ideals of $R$.*

**Proposition 1.2.8.** *([9]) Let $M$ be an $R$- module. The following conditions are equivalent.*

$(i)$ *Any non-empty collection of submodules of $M$ has a maximal element.*

$(ii)$ *For any increasing sequence of submodules of $M$, $M_1 \subset M_2 \subset ... \subset M_n \subset ...$, there exists some integer $m$ such that $M_k = M_m$ for all $k \geq m$.*

$(iii)$ *Every submodule of $M$ is finitely generated.*

**Definition 1.2.9.** ([9]) An $R$-module $M$ is called *Noetherian* if it satisfies any one of the above equivalent conditions.

**Definition 1.2.10.** ([9]) A ring $R$ is called a *Noetherian ring* if the $R$-module $R$ is a Noetherian.

**Example 1.2.9.** $(i)$ Any field $R$ is a Noetherian ring.

$(ii)$ Any finite ring $R$ is Noetherian.

$(iii)$ Any principal ideal ring is Noetherian.

**Definition 1.2.11.** ([13]) An integral domain $R$ is a *Dedekind domain* if every proper ideal of $R$ is a product of prime ideals.

**Example 1.2.10.** $(i)$The class of Dedekind domain is precisely the class of Noetherian Prüfer domains.

$(ii)$ The ring $\mathbf{Z}[\sqrt{5}] = \{a\ +\ b\sqrt{5}|\ a,\ b \in \mathbf{Z}\}$ is a Dedekind domain.

There are large number of equivalent conditions for a Noetherian integral domain to be a Dedekind domain. The equivalence of some of these conditions are stated in the following theorem.

**Theorem 1.2.11.** *([13, Theorem 6.20]) If $R$ is a Noetherian integral domain, then the following statements are equivalent:*

$(i)$ *$R$ is a Dedekind domain;*

$(ii)$ *Every nonzero ideal of $R$ generated by two elements is invertible;*

$(iii)$ *For every maximal ideal $P$ of $R$, the ring of quotients $R_P$ is a valuation ring;*

$(iv)$ *$(A+B):C\ =\ (A:C)\ +\ (B:C)$ for all ideals $A,\ B,\ C$ of $R$;*

$(v)$ *$C:(A\cap B) = (C:A)\ +\ (C:B)$ for all ideals $A,\ B,\ C$ of $R$;*

$(vi)$ *If $P$ is a maximal ideal of $R$, then every $P$-primary ideal of $R$ is a power of $P$.*

$(vii)$ *If $P$ is a maximal ideal of $R$, then the set of $P$-primary ideals of $R$ is totally ordered by inclusion.*

## 1.3 Dedekind Modules

The concept of an invertible submodule was introduced in [5] as a generalization of the concept of an invertible ideal and several authors have extensively studied invertible submodules and Dedekind modules.

Following Al-Alwan and Naoum [5]

**Proposition 1.3.1.** *([5]) Let $M$ be an $R$-module and let $S$ be the set of non-zero divisors of $R$. Then $T = \{t \in S : tm = 0 \ for \ some \ m \in M \ implies \ m = 0\}$ is a multiplicative set.*

**Definition 1.3.1.** ([5]) Let $T^{-1}R$ be the localization of $R$ at $T$ in the usual sense and let $N$ be any submodule of $M$. If $x = \frac{r}{t} \in T^{-1}R$ and $n \in N$, then we say that $xn \in M$ if there is a $m \in M$ such that $tm = rn$.

**Definition 1.3.2.** ([5]) A nonzero submodule $N$ of $M$ is *invertible* if $N'N = M$, where $N' = \{x \in T^{-1}R : xN \subseteq M\}$ and $M$ is a *Dedekind module* provided that $M$ is nonzero and each nonzero submodule of $M$ is invertible.

Note that $N'$ is an $R$-submodule of the quotient field $K$.

Furthermore, $M$ is said to be a $D_1$*-module* if each non-zero cyclic submodule of $M$ is invertible. It is clear that every Dedekind module is a $D_1$-module; however the converse is false.

**Example 1.3.2.** $(i)$ The Prüfer group $\mathbf{Z}_{p^\infty}$ has no proper invertible submodule.
$(ii)$ Let $R$ be a ring, then $R$ is a $D_1$-module if and only if $R$ is an integral domain.
$(iii)$ $\mathbf{Q}$ is a Dedekind $\mathbf{Z}$-module.