Allameh Tabatabaei University

E.C.O. College of Insurance

Applications of phase type models in modeling Mortality

Submitted in Partial Fulfillment of the Requirements for

the Degree of MS in Actuarial Science

By:

Zahra Shojaei Azar

Supervisor:

Dr. Amin Hassan Zadeh

Advisor:

Dr.Ghadir Mahdavi

Iran-Tehran

September-2012

To

Two angels of my life

My Dear father

And

Kind mother

Who have been the best supporters

for me in the all moments of my life

Acknowledgements

I would like to appreciate Dr. Amin Hassanzadeh, the honorable supervisor and Dr. Ghadir Mahdavi the respectful advisor of the present thesis, because of their unsparing efforts to guide me, devoting a lot of time, and their patience during performance of this thesis.

I also appreciate the referec of the dissertation, Dr. Reza Ofoghi because of the time he spent on studying my thesis and for constructive comments.

I deem necessary to appreciate all my friends who helped me, special H. Fathizadeh and Leili Dibamehr.

Abstract

In this study, we will introduce Markov Chain and describe its applications. A phase type random variable is defined as the time until absorption in a continuous Markov Chains. We will use phase type distribution in modeling mortality. In this project, instead of real age we will use physiological age, which is defined as health index, and then by matrix analytic method in the phase type context, we will compute all probabilities related to mortality of real age. Next, we consider each status of Markov Chain as a physiological age and status of absorption in the Markov Chain as the status of death. After modeling, we will estimate the parameter of the phase type distribution by the least squares method then, we will fit the model to life table of Iran. Eventually, using these parameters, we will compute closed-form expressions for the actuarial present values of the whole life insurance and whole life annuity.

Keywords: Phase-type Distribution, Markov Chains, Mortality Modeling, Annuities Pricing, Life Table, Aging Process, Physiological Ages.

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Chapter1: Introduction

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1.4. Literature review of phase type distributions

1.1. Introduction

For doing calculations related to insurance and investigating long term liabilities of social security organization and other retirement funds or for doing population predicting, actuaries need to use statistics and reliable information. One of the most important and effective information in this field is mortality table. Using mortality tables is one of the oldest and most common ways for measuring and investigating mortality rates in a population. Thus, can be said mortality tables are important tools for actuaries to describe mortality pattern in a population.

In the life insurance with building life table, which indicating mortality and people's survival probability in different ages, we try to know probable pattern of payments. When there is no mathematical model, a mortality table is just a report from (on the population) past experiments, and applying it in the future events needs to hope that past experiment will repeat unchanging. Therefore, there have been many efforts to find a model to create these tables. Undoubtedly, obtaining a mathematical model for finding probability of mortality is helpful. It is clear that, working with function which has a few parameters, and also, has a uniform diagram is much easier than working with life tables include many parameters.

Also, in many cases it is needed to interpolation, extrapolation, flattening, and other several statistical adjustments in order to find the probabilities of death which are used widely in calculations of technical premium. All of the models have used, cannot determine distribution of death time explicitly, thus creating a model can determine death time by fitting data seems helpful.

In this study, we propose a hypothetic modeling framework using a finite-state continuous-time Markov process with a single absorbing state (death) to describe a physiological aging process of a human body. Then, we present our fitting result for the Iran data. Later, we use the usefulness of the model and the matrix-analytic method in pricing insurances and annuities.

In this chapter, the importance of application phase type distribution of mortality modeling as well as the literature review will be reviewed. In the second chapter, we will represent a few of survival modelling used and in the third chapter; we will go through the details of Markov chain and phase type distribution and will show detail estimate parameters in the chapter four, also we will provide the empirical analysis for Iran mortality life table. Finally in chapter five the result and recommendations will be represented.

1.2. Importance of application phase type distribution of mortality modeling:

Mortality rates are important risk factors in both life insurance and pensions, which affect fair values, premium rates, and risk reserves are calculated.

A life table is the fundamental tool to describe age pattern of mortality for actuaries but, when there is no suitable mathematical law of mortality available, a life table can be just an alternative. Actuaries have used two below mortality models widely:

1-Hellgman-Pollard model in actuarial science

2 -Lee- Carter model in survival analysis

Nevertheless, in these models, we cannot determine the distribution of the time until death explicitly, i.e. there is not any analytical method for both models. Hence, actuaries are supposed to focus on numerical or statistical methods, while using these models. Here we will study on a model in which time until death is explicit and has a phase type distribution.

The advantage of using this model is having closed-form expressions for many quantities used in life insurance and annuity calculations. These closed-form expressions provides easier calculation of whole life insurance and whole life annuity. In last, a simple analytical survival function will have fewer parameters to fit the mortality data, while, the Lee- Carter model may need more than 300 parameters to fit mortality data (Lin and Liu, 2007).

1.3. Literature review of Mortality Modeling:

Human have been trying to understand life and death for as long as they existed. Every culture and nation has legends about the origins and reasons for being born and dying. Scholars would formulate these ideas into questions about whether or not human survival is governed by some law, and if so, what it is and how science can explain it. The obvious source of information on this topic is birth and death data.

A life table is an indispensable component of many models in actuarial science. In fact, some scholars fix the date of the beginning of actuarial science as 1693. In that year, famous English astronomer Edmund Halley published “An Estimate of the Degrees of the Mortality of City Breslau”. The life table, called the Breslau Table, contained in Halley’s paper remains of interest because of its surprisingly modern notation and ideas. Soon after, in 1740, the earliest life tables for males and females were published by Nicholas Struyck (Pitacco, 2003). Around this time, some mathematicians became interested in modeling survival too; as a result, Abraham De Moivre produced the first known analytic model for probability of survival as a linear function of the person’s current age. However, De Moivre recognized that his model failed to give accurate representation of human survival across all ages.

So the search for a better model continued, and in 1825 Benjamin Gompertz presented his version of the formula for survival probability, based on his recognition that human mortality displayed some exponential patterns for most ages. His result is believed to be the most influential parametric mortality model the literature. Some years later, in 1860, Makeham noticed that Gompertz's model was not adequate for higher ages and amended it in an effort to correct this deficiency (Higgins, 2003). Despite developments after 1860, Gompertz's and Makeham's models remain to this day among the most popular choices for mortality models.

In the early 20th century, Italian economist and sociologist Vilfredo Pareto put forth his idea for a model of mortality after working as an engineer and studying the social problems of the day. In the 1939's, Wallodi Weibull was devising a model to predict time until next failure of a technical system, which was later adapted as a survival model with human organs seen as technical parts that eventually fail. Throughout the last century, there were other contributions but most of them were modifications/generalizations of those of Gompertz and Makeham.

In recent decades, the study of mortality has become more complex and modern. Due to ever-increasing computational capacities, parametric models may involve up to ten parameters (such as the model (Heligman and Pollard, 1980) with eight), or utilize parameters dependent on current year as well as the person's age (Lee-Carter model from 1992). The newest direction in the study of human survival is the idea of modelling mortality as a stochastic process (see Yashin, 2001), but, as noted by Higgins (2003), the development of such models is in its infancy stage.

1.4. Literature review of phase type distribution:

We start in this section, with a short bibliographic review. Since their introduction by Neuts (1975), phase type (PH) distributions have been used in a wide range of stochastic modeling applications in areas as diverse as telecommunications, finance, teletraffic modelling, biostatistics, queueing theory, drug kinetics, reliability theory, and survival analysis. Though phase- type distribution can be traced back to the pioneering work of Erlang (1909) and to jensen (1953). Erlang in 1917, was the first person to extend the exponential distribution with his “method of stages”. He defined a nonnegative random variable as the time taken to more through a fixed number of stages (or states), spending an exponential amount of time with a fixed rate in each one. Nowadays we refer to distributions defined in this manner as Erlang distributions.

Neuts (1981) generalized Erlang’s method of stages by defining a PH random variable as the time spent in the transient states of a finite-state continuous-time Markov Chain with one absorbing state, until absorption.

Prior to Neuts’s work much of the research in stochastic modeling and queueing theory relied on random variables of interest and service times being modeled by the exponential or Erlang distribution, and point and interarrival by the poisson or Erlang renewal processes.

PH distributions constitute a much more useful class of distributions for a number of reasons. First, they form a versatile class of distributions that are dense in the class of all distributions defined on the nonnegative real numbers. That is, they can approximate any nonnegative distribution arbitrarily closely, although the number of states needed may be large. Second, since they have a simple probabilistic interpretation in terms of continuous-time Markov chains, they exhibit a Markov structure which enables an easier analysis of models that use them instead of general distributions. Lastly, the use of PH distributions in stochastic models often enables algorithmically tractable solutions to be found. If PH distributions are used, many quantities of interest that are used in algorithms to compute performance measures can be expressed simply in terms of the inverse and exponential of matrices that contain only real entries. Also, since the class of PH distributions is closed under a variety of operations, such as finite mixtures and convolutions, systems with PH inputs often have PH outputs. In bellow we will be describe some studies done about phase type distribution.

O'Cinneide (1990) studied theoretical properties of phase-type distributions, such as their characterization. He said that a distribution with rational Laplace-Stieltjes transform is of phase type if and only if it is either the point mass at zero, or it has a continuous positive density on the positive real parts and its Laplace-Stieltjes transform has a unique pole of maximal real part (which is therefore real). This result is proved, and the corresponding characterization of discrete phase-type distributions is stated and proved. Her methods are based on a geometric property of the set of phase-type distributions associated with a Markov chain.

Aalen (1995), introduced phase type distributions based on Markov processes for modelling disease progression in survival analysis. For tractability and to maintain the Markov property, these use exponential waiting times for transitions between states. He considered to finite-state, time-continuous, homogeneous Markov chain with an absorbing state. According to Aalen, the time to absorption in such a state is said to have a phase-type distribution. Markov models are useful because they may incorporate an understanding of how a phenomenon, for example, a disease, moves through different stages. Densities and hazard rates of phase-type distributions can be calculated from standard Markov chain theory.

Hazard rates are usually asymptotically constant due to quasi-stationarity; that is, conditional on non absorption, the distribution on the transient space converges to a limiting distribution. Quasi- stationarity can also give an explanation of the various shapes of hazard rates that may occur.

Riska et al. (2002), proposed a new technique for fitting long-tailed data sets into phase-type (PH) distributions. The proposed method is accurate and computationally efficient. Furthermore, it allows one to apply existing analytic tools to analyze the behavior of queueing system long-tailed arrival or systems with long-tailed arrival.

Steve Drekic et al. (2004), studied the distribution of the deficit at ruin in Sparr Anderson renewal risk model given that ruin occurs, and showed that if individual claim amounts have a Phase type distribution, it will be a simple Phase type representation for the distribution of the deficit.

Bladt (2005), studied on phase type distributions and retrieved some of their basic properties through appealing probabilistic arguments which, indeed, constitute their main feature of being mathematically tractable. Then, he provided the necessary background on the theory of Markov jumps process in order to introduce the concept of phase type distribution. According to Bladt, any positive distribution may be approximated arbitrarily closely phase type distributions whereas exact solutions to many complex problems in stochastic modeling can be obtained either explicitly or numerically. Then, Bladt illustrated in an example where he calculated the ruin probability for a rather general class of surplus process where the premium rate is allowed to depend on the current reserve and where claims sizes are assumed to be of phase type.

Panchenko and Thummler (2007), studied that approximating the empirical distribution of a measured data trace by a phase-type distribution has significant applications in the analysis of stochastic models. They showed how to effectively use the aggregated trace within a PH fitting approach because elements of a large traffic trace can be aggregated to a smaller number of 50–200 weighted elements, such that traces with ten million elements can be accurately fitted in a few seconds.

Fackrell (2008), studied applications of PH distributions in the healthcare sector. His results are as the following:

1. A more general PH representations should be considered for fitting length of stay and interarrival time data because of the extra flexibility.
2. More sophisticated models that use general PH distributions should be considered in modeling healthcare systems.
3. The wider literature on matrix-analytic methods should be consulted when developing stochastic models used in modeling healthcare systems.

Reinecke and Wolter (2008), studied the application of phase-type distribution inWeb-Services based Service-Oriented Architectures (SOAs) become ever more important. The Web Services Reliable Messaging (WSRM) standard provides a reliable messaging layer to these systems. They applied the parameters of acyclic continuous phase-type (ACPH) approximations for message transmission times in a Web Services Reliable Messaging implementation confronted with several different levels of IP packet loss. These parameters illustrate how large data sets may be represented by just a few parameters. Also, they showed that the ACPH approximations presented in her paper can be used for the stochastic modeling of SOA system.

Hassan zadeh et al. (2010) , applied phase-type models in actuarial calculations for disability insurance. They demonstrated that the changes in status of disability insured can be appropriately captured by a phase-type model. They showed that Using such a model, explicit and easily calculable expressions are obtained for relevant probabilities and actuarial present values.

Although phase type distributions are applied widely in different fields, up to now, it have not been any significant work about the application of phase type distribution in modeling of mortality formulas. So in this part we referred briefly to the applications of phase type distributions in other different fields.

Chapter 2: Mortality Modeling

2.1. Survival Distribution

2.2. Life Table

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2.5.1. Stochastic models

2. Mortality Modeling

The life insurance and annuity contracts that were the object of study of the early actuaries were very similar to the contracts written up to the 1980s in all developed insurance markets. Hence, the lifetime random variable X and its associated mortality model are the basic building blocks in actuarial mathematics. In this section, we first introduce basic concepts and actuarial notation related to mortality modeling then, we present empirical results on people mortality trends, next will describe dynamic models briefly.

2.1. Survival Distributions

Let us consider a newborn child, for this propose We begin by considering a continuous age-at-death variable X. specifically, X is a nonnegative random variable representing the lifetime of an newborn in a cohort or population.

All distribution functions related to the random variable X, unless stated other-wise, are defined over the interval [0, ∞). Let f(x) denote the probability density Function (p.d.f.) of X and let the cumulative distribution function (c.d.f.) be

$$F(x) = P(X \leq x) = \int_0^\infty f(t)dt \qquad \text{(2-1-1)}$$

Then F(x) represents the probability that (x) does not survival beyond age x. The probability of an individual surviving to age x is given by the survival function

$$s(x) = 1 - F(x) = P(X > x) = \int_x^\infty f(t)dt. \qquad \text{(2-1-2)}$$

The function s(x) is called the survival function (s.f). For any positive x, s(x) is the probability a newborn will attain age x. The distribution of x can be defined by specifying either the function F(x) or the function s(x). Within actuarial science and demography, the survival function has traditionally been used as a starting point for further developments. Within probability and statistic, the d.f usually plays this role. However, from the properties of the d.f, we can deduce corresponding properties of the survival function.

Using the laws of probability, we can make probability statements about the age-at-death in terms of either the survival function or the distribution. For example, the probability that a newborn dies between ages x and z (x<z) is $\Pr(x < X < z) = F_X(z) - F_X(x)$

$$= s(x) - s(z). \qquad (2\text{-}1\text{-}3)$$

Also, the conditional probability that a newborn will die between the ages x and z, given survival to age x, is

$$\Pr(x < X \leq z | X > x) = \frac{F_X(z) - F_X(x)}{1 - F_X(x)} \qquad (2\text{-}1\text{-}4)$$

But, a very important concept in mortality modeling is the force of mortality (often referred to as the hazard function or hazard rate or failure rate in other fields such as in reliability theory), which is defined as:

$$\mu(x) = \lim_{\Delta x \to 0} \frac{(x < X \leq x + \Delta x | X > x)}{\Delta x} \qquad (2\text{-}1\text{-}5)$$

$$= \lim_{\Delta x \to 0} \frac{F_X(\Delta x + x) - F_X(x)}{(1 - F_X(x))\Delta x} = \frac{f(x)}{1 - F_X(x)} = \frac{f(x)}{s(x)} \qquad (2\text{-}1\text{-}6)$$

The force of mortality specifies the instantaneous rate of death at age x, given that the individual survives up to age x.

Any one of the function f(x), F(x), s(x), or μ(x) can be used to specify the distribution of X. It is easy to see that, given an expression for any one of the above four functions, the other three can be derived. For example, for newborn child in terms of the force of mortality μ(x), we have:

$$s(x) = e^{-\int_0^x \mu(t)dt} \tag{2-1-7}$$

$$F(x) = 1 - s(x) = 1 - e^{-\int_0^x \mu(t)dt} \tag{2-1-8}$$

And

$$f(x) = \mu(x)e^{-\int_0^x \mu(t)dt} = \mu(x)\ {}_xp_0 \tag{2-1-9}$$

That, ${}_xp_0 = P[(0) will\ attain\ t0\ x\ year]$,

We will often make use of the future lifetime random variable (or residual lifetime)$T(x)$. $T(x)$ is the time-until-death variable measured from the date that a contract has been issued to an individual of age x. In the following, the symbol (x) is used to denote a life-aged-x. The distribution function for $T(x)$can be derived from the distribution function for X. In actuarial science, special symbols have been assigned to denote the distribution function for T as follows.

Actuarial Notation:

For t ≥ 0, we define

$${}_tq_x = P(T(x) \le t) = P(x < X \le x + t | X > x) = \frac{s(x)-s(x+t)}{s(x)} \tag{2-1-10}$$

$${}_tp_x = P(T(x) > t) = P(X > x + t | X > x) = \frac{s(x+t)}{s(x)} \tag{2-1-11}$$

The symbol ${}_tq_x$ can be interpreted as the probability that (x) will die within t years; that is, ${}_tq_x$is the c.d.f of T(x). Similarly, ${}_tp_x$ can be interpreted as the