IN THE NAME OF GOD

THE COMMUTANT OF MULTIPLICATION BY z ON THE CLOSURE OF POLYNOMIALS IN $L^t(\mu)$

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To:

my dear parents

and

memory of Prof. K. Seddighi

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ABSTRACT

THE COMMUTANT OF MULTIPLICATION BY z ON THE CLOSURE OF POLYNOMIALS IN $L^t(\mu)$

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In this thesis μ represents a positive, compactly supported Borel measure in the complex plane. For each t in $[1, \infty)$, the space $P^t(\mu)$ consists of the functions in $L^t(\mu)$ that belong to the (norm) closure of the analytic polynomials in one variable. Multiplication by z on $P^t(\mu)$ is a bounded operator, which is denoted by S_μ. Throughout, it is assumed that S_μ is irreducible. J. Thomson in [T] has shown that the set of bounded point evaluations, $bpe(P^t(\mu))$, is a nonempty simply connected region G having the following properties: (a) The spectrum of S_μ, denoted $\sigma(S_\mu)$, equals $\bar{G}$; (b) The essential spectrum of S_μ, denoted $\sigma_e(S_\mu)$, equals ∂G; (c) The commutant of S_μ, identified in the coustomary way with $P^t(\mu) \cap L^\infty(\mu)$, is isometrically isomorphic to the bounded analytic functions on G, denoted by $H^\infty(G)$. We let $\sim: f \longrightarrow \tilde{f}$ signify this Banach algebra isometric isomorphism from $H^\infty(G)$ to $P^t(\mu) \cap L^\infty(\mu)$. Let φ be a Riemann map from G to where $\mathbb{D}$ where $\mathbb{D}$ denotes the open unit disc, it follows that $\tilde{\varphi}$ is in $P^t(\mu) \cap L^\infty(\mu)$. Now, define a measure, ν, by setting $\nu = \mu \circ \tilde{\varphi}^{-1}$. A routine argument shows

that S_ν acting on $P^t(\nu)$ is irreducible with $bpe(P^t(\nu)) = \mathbb{D}$. If ψ is the inverse of φ, then we also have that $\tilde{\psi}$ is in $P^t(\nu) \cap L^\infty(\nu)$.

In chapter I, a historical review of the theory and some related backgrounds of it are given. In chapter II, we will investigate the structure of $P^t(\mu)$, relate the structure of $P^t(\mu)$ to bounded and analytic bounded point evaluations and provide a result relating the above classes of operators to cyclic subnormal operators. In chapter III, the measure μ restricted to the boundary of G is absolutely continuous with respect to the harmonic measure on G and the function $\tilde{\psi}$ is almost a one-to-one map from a carrier of $\nu \mid_{\partial\mathbb{D}}$ to a carrier of $\mu \mid_{\partial\mathbb{D}}$ are proven. In chapter IV, it is proved that the space $P^2(\mu) \cap C(supp(\mu)) = A(G)$ where $C(supp(\mu)$ denotes the continuous functions on $supp(\mu)$ and $A(G)$ denotes those functions continuous on $\bar{G}$ that are analytic on G.

TABLE OF CONTENT

CONTENT PAGE

CHAPTER I:	**Introduction**	**1**
1.1.	Historical Review of the Theory	2
1.2.	Related Topics from Measure Theory	6
1.3.	The Support of a Measure	8
1.4.	Harmonic Measure and Related Topics	11
1.5.	Vitushkin Localization Operators	17
CHAPTER II:	**Bounded Point Evaluations For $P^t(\mu)$ Spaces and the Structure of such Spaces**	**19**
2.1.	Bounded Point Evaluations for $P^t(\mu)$ Spaces	21
2.2.	Analytic Bounded Point Evaluations for $P^t(\mu)$	30
2.3.	The Structure of $P^t(\mu)$	39
2.4.	Applications to Subnormality and Operator Theory	41
CHAPTER III:	**On the Commutant of S_μ and Comparing some Function Properties on it**	**46**
3.1.	Publling S_μ Back to the Disc	48
3.2.	Boundary Behavior of the Measure	59
3.3.	Nontangential limits of Functions in $P^t(\mu) \cap L^\infty(\mu)$	68

CHAPTER IV: $P^2(\mu) \cap C(supp(\mu))$ **As a Disc Algebra over** $bpe(P^2(\mu))$ **73**

Appendix **85**

5.1. In the Honor of Prof. Thomson 85

REFERENCES **90**

ABSTRACT AND TITLE PAGE IN PERSIAN

CHAPTER I

INTRODUCTION

1. INTRODUCTION

1.1. Historical Review of the Theory

Selfadjoint and normal operators were the first classes of Hilbert space operators to be analyzed. The prototype is multiplication by the coordinate function z on $L^2(\mu)$, where μ is a compactly supported measure on the complex plane. If we take μ to be the arc-length measure on the unit circle, we obtain the shift operator on two-tailed square-summable sequences. The spectral theorem asserts that any normal operator can be represented as a direct sum of such multiplication operators on L^2-spaces. The theory has the flavor of measure and integration.

An operator S is defined to be *subnormal* if it is the restriction of a normal operator to an invariant subspace. These operators, introduced in 1950 by P. Halmos, have as prototype the multiplication by z on the closure $P^2(\mu)$ in $L^2(\mu)$ of the analytic polynomials. In the case of arc-length measure we obtain multiplication by z on the Hardy space $H^2(d\theta)$, which is equivalent to the shift operator on one-sided square-summable sequences. Now in addition to measure theory there is a strong taste of function theory. One asks, when are the polynomials in z dense in $L^2(\mu)$? If they are not dense, can the defect be accounted for by the analyticity of functions in $P^2(\mu)$ on some nonempty open subset of the plane?

Function theory enters the picture through the functional calculus. Let K be a compact subset of the complex plane containing the spectrum $\sigma(S)$ of S. If f is a rational function with poles off K, then $f(S)$ is defined, and the operator norm of $f(S)$ is dominated by the supremum norm of f over K. Thus we obtain an algebra homomorphism of the uniform algebra $R(K)$ into $\mathcal{B}(\mathcal{H})$. The later space is a dual space, and if μ is a measure on K with sufficiently ample support, the operator calculus extends to a weak-star continuous homomorphism from the weak-star closure $R^\infty(K,\mu)$ of $R(K)$ in $L^\infty(\mu)$ into $\mathcal{B}(\mathcal{H})$. For $K = \sigma(S)$ and special μ, we have even an isometric weak-star homeomorphism. Information about $R^\infty(K,\mu)$ yields information about S via this functional calculus. The functional calculus also extends to the bidual $R(K)^{**}$, which is an inverse limit of the spaces $R^\infty(K,\mu)$.

The problem of uniform approximation by analytic polynomials was solved in 1953 by S. N. Mergelyan. The uniform limits on K of polynomials in z are precisely the functions in $C(K)$ which extend continuously to be analytic on the interior of the polynomial hull of K (the union of K and the bounded components of the complement of K). Mergelyan's proof was completely constructive. He used the Cauchy-Green formula to split the singularities into bite-sized chunks. This localization technique and the Cauchy transform permeate the constructive side of the theory. In 1963 a semiabstract proof of Mergelyan's theorem was obtained through the efforts of E. Bishop, L. Glicksberg, and J. Wermer. Concrete function theory still played an important role. The abstract theory kicks in once one knows that $P(K)$ is a Dirichlet algebra, and this requires an (easier) approximation theorem for harmonic functions.

The L^2-approximation problem remains a difficult nut to crack, with complete results only in very special cases. On the other hand, the weak-star approximation problem turned out to be more tractable. In 1972, through a beautiful analysis, D. Sarason obtained a decomposition of $P^\infty(\mu)$ as a direct sum of $L^\infty(\mu_s)$ and $P^\infty(\mu_a)$, where the latter summand is isometric and weak-star homeomorphic to the algebra $H^\infty(U)$ of bounded analytic functions on a certain open set U. Sarason's proof is constructive in nature. He constructs an increasing chain of Dirichlet algebras $R(K_\alpha)$, indexed by the ordinals, such that functions in a dense subset of each can be approximated appropriately by bounded sequences of functions from the predecessors. The procedure terminates long before the first uncountable ordinal is reached, and U is the interior of the final (smallest) K_α. Sarason's analysis had a substantial impact on the study of subnormal operators. The functional calculus that was obtained played a role in S. Brown's proof in 1978 of the existence of invariant subspaces for subnormal operators. In turn, this theorem attracted a lot of attention to the theory. In 1981 Conway published a research monograph in the red Pitman series that described this theory and culminated in Sarason's analysis of $P^\infty(\mu)$ and Brown's invariant subspace theorem.

The next natural step was to apply the tools of rational approximation theory, specifically properties of $R^\infty(K,\mu)$, to obtain information on subnormal operators. The problem of uniform approximation by rational functions had been solved in 1967 by A. G. Vitushkin in terms of analytic capacities.

In 1972, A. M. Davie obtained a striking result, a decomposition of the bidual $R(K)^{**}$ as a direct sum of an L^∞-space and an algebra isometric and weak-star homeomorphic to $R^\infty(K, \lambda_Q)$, where λ_Q is the area measure restricted to the set Q nonpeak points of $R(K)$. The set Q has full area density at each of its points, and in some respects it behaves as a finely open set does with respect to bounded harmonic functions. Intuitively the algebra $R^\infty(K, \lambda_Q)$ can be thought of as an algebra of bounded analytic functions on Q. In the case $R(K)$ is a Dirichlet algebra, Q is just the interior of K.

The next step was taken in 1974 by J. Chaumat, who gave a similar description of $R^\infty(K, \mu)$ for an arbitrary measure μ on K. The set playing the role of Q, denoted by $E(\mu)$, consists of the points $z \in K$ for which the evaluation functional is weak-star continuous on $R(K)$ in the weak-star topology of L^∞ of the measure μ deprived of its mass at $\{z\}$. Again $E(\mu)$ has full area density at each of its points, and $R^\infty(K, \mu)$ can be decomposed as a direct sum of an L^∞ term and an algebra isomorphic to $R^\infty(K, \lambda_{E(\mu)})$. In 1985 Cole and Gamelin gave a proof of Chaumat's theorem along the constructive lines of Sarason's proof, producing a chain of intermediate algebras which are invariant under the T_φ-operators used by Vitushkin to split singularities. (In a weak and uninspired moment someone dubbed these "T-invariant" algebras.) The theory so unified covers any algebra invariant under the localization operators.

Recently J. Thomson made a breakthrough on the L^2 polynomial approximation problem. He succeeded in answering an old question on the existence of analytic point evaluations, showing that if $P^2(\mu) \neq L^2(\mu)$ then there is a nonempty open set U on which the functions in $P^2(\mu)$ are analytic. This is a seminal result, which will have substantial spin-off. The basic idea of the proof was inspired by an earlier proof technique of M. S. Melnikov, who had shown in 1976 that each Gleason part for $R(K)$ is "area connected".

1.2. Related Topics from Measure Theory

Suppose f is a mapping from X into Y, $\mathcal{F}$ is a σ-field in X and $\mathcal{S}$ is a σ-field in Y, then we say that f is a measureable transformation from $(X, \mathcal{F})$ into $(Y, \mathcal{S})$ if $f^{-1}(E) \in \mathcal{F}$ for every E in $\mathcal{S}$. This condition can also be written $f^{-1}(\mathcal{S}) \subset \mathcal{F}$.

In the case of measurable functions, these are measurable transformations from $(X, \mathcal{F})$ to $(\mathbb{R}^*, \mathcal{S})$ in which $\mathcal{S}$ is the σ-field of Borel sets in $\mathbb{R}^*$ (extended real numbers.) Given mappings $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ we can consider the composition $g(f) : X \longrightarrow Z$ defined by $g(f)(x) = g(f(x))$. In particular, if $g : Y \longrightarrow \mathbb{R}^*$ is an extended real-valued function on Y, then $g(f)$ defines an extended real function on X.

Lemma 1.2.1. If $f : X \longrightarrow Y$ is a measurable transformation from $(X, \mathcal{F})$ into $(Y, \mathcal{S})$ and $g : Y \longrightarrow \mathbb{R}^*$ is $\mathcal{S}$-measurable as a function with extended real values, then the composition $g(f)$ is $\mathcal{F}$-measurable.

Proof. For any Borel set B in $\mathbb{R}^*$ we have

$$\begin{aligned} \{x : g(f)(x) \in B\} &= f^{-1}\{y : g(y) \in B\} \\ &= f^{-1}(E) \text{ for some } E \in \mathcal{S}, \end{aligned}$$

and is therefore in $\mathcal{F}$. □

Remark. We have obtained a special case of this lemma when we want to prove that a Borel measurable function of a measurable function is measurable.

If we start with a measure space $(X, \mathcal{F}, \mu)$ and f is a measurable transformation from $(X, \mathcal{F})$ into $(Y, \mathcal{S})$ it is natural to use f to define a measure ν on $\mathcal{S}$ by putting

$$\nu(E) := \mu(f^{-1}(E)) \quad \text{for} \quad E \in \mathcal{S}. \tag{1}$$

With this definition of ν it is immediate that $(Y, \mathcal{S}, \nu)$ is a measure space. If (1) holds we will write $\nu = \mu f^{-1}$. This allows us to carry out a "change of variable" in an integral.

Theorem 1.2.2. Suppose f is a measurable transformation from a measure space $(X, \mathcal{F}, \mu)$ to a measurable space $(Y, \mathcal{S})$ and $g : Y \longrightarrow \mathbb{R}^*$ is $\mathcal{S}$-measurable then

$$\int_Y g d(\mu f^{-1}) = \int_X g(f) d\mu$$

in the sense that if either integral exists so does the other and the two are equal.

Proof. See [Ty, Thm. 6.8]. □

Remark. If, in the notation of Theorem 1.3.2,F is a measurable subset of Y, then an application of Theorem 1.3.2 to the function $\chi_F g$ yields the relation

$$\int_F g d\mu f^{-1} = \int_{f^{-1}(F)} g f d\mu.$$

Sometimes in integration, when the variable is changed, one wants to integrate with respect to a new measure $\nu \neq \mu f^{-1}$. We can do this easily when μf^{-1} is absolutely continuous with respect to ν.

Theorem 1.2.3. Given σ-finite measure space $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{S}, \nu)$ and a measurable transformation f from $(X, \mathcal{F})$ into $(Y, \mathcal{S})$ such that μf^{-1} is absolutely continuous with respect to ν

$$\int g(f) d\mu = \int g.\phi d\nu,$$

where ϕ is the Radon-Nikodym derivative $\frac{d(\mu f^{-1})}{d\nu}$, for every measurable $g : Y \longrightarrow \mathbb{R}^*$ in the sense that, if either integral exists, so does the other and the two are equal.

Proof. See [Ty, Thm. 6.9]. □

Remark. It is clear from the above that the function ϕ plays the part of Jacobian (or rather the absolute value of the Jacobian) in the theory of transformations of multiple integrals. In general it is not easy to obtain an explicit value for the Radon-Nikodym derivative $\frac{d(\mu f^{-1})}{d\nu}$, but in important special case this can be done.

1.3. The Support of a Measure

Let X be a topological space and $\mathcal{B}$ be the Borel sets of X. The support of a measure $\mu : \mathcal{B} \longrightarrow [0, +\infty]$, if it exists, is a closed set, denote $supp(\mu)$, satisfying:

1. $\mu((supp(\mu))^c) = 0$;
2. If G is open and $G \cap supp(\mu) \neq \phi$, then $\mu(G \cap supp(\mu)) > 0$.