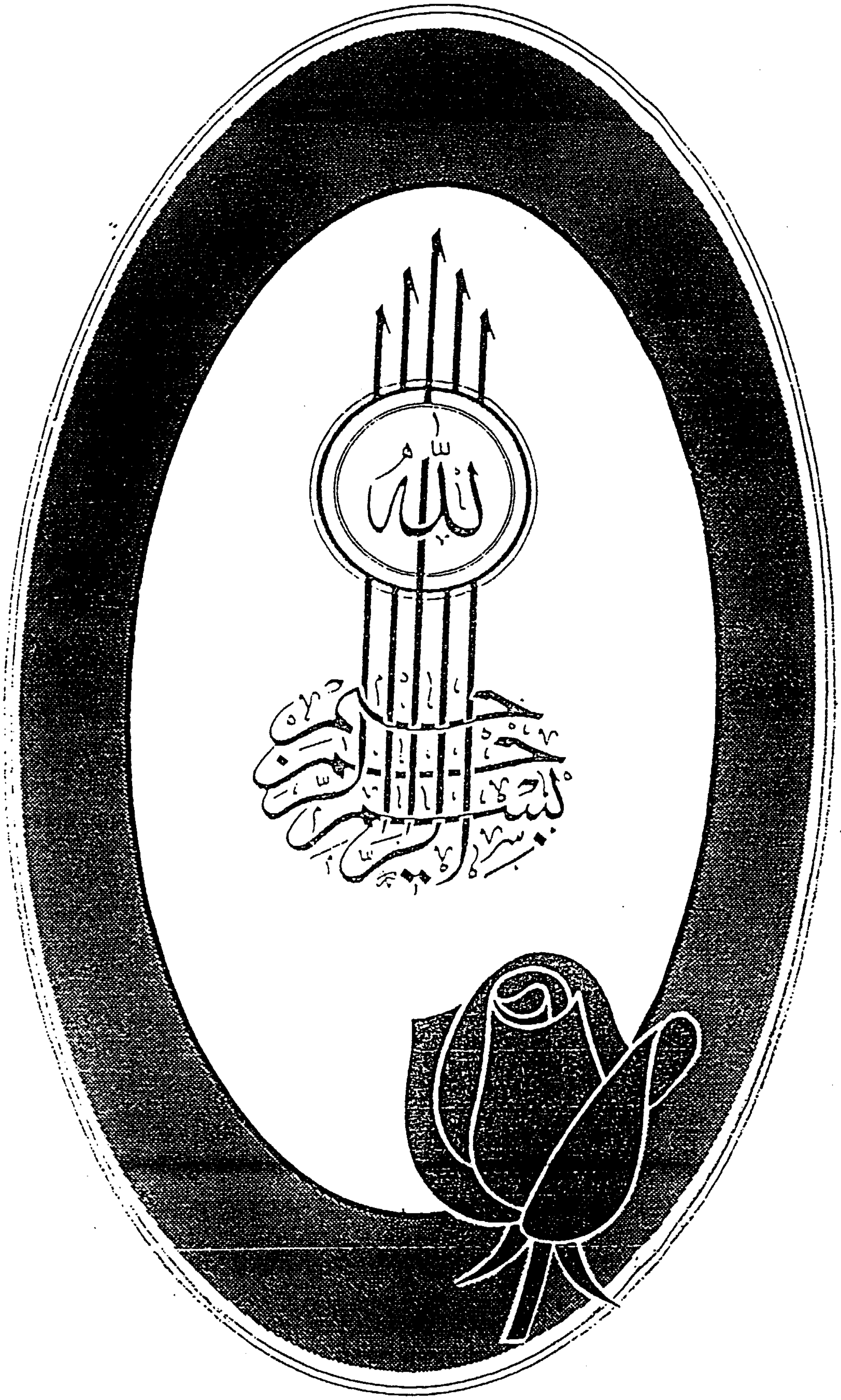
الله
بسم الله الرحمن الرحيم

In the name of God

Shahid Bahonar University of Kerman

Faculty of Mathematics and Computer Science

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Hyper BCK and K-algebras

BY

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Abstract

In this thesis first we give some informations about BCK-algebras and hyper algebraic structures. Then we introduce the notions of hyper BCK-algebra (which is a generalization of a BCK-algebra), hyper BCK-ideal, hyperK-algebra and positive implicative hyper K-ideals of types 1,2,3 and 4. Also we give many examples to show that these notions are different. Finally we prove some theorems and obtian some results about the above mentioned concepts. In particular we give some classifications of hyperK-algebras of order 3, which satisfy the normal or simple conditions.

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Contents

Chapter 1

Introduction and Preliminaries

1.1 Introduction and Preliminaries

The notion of BCK-algebra was formulated first in 1966 by K. Iseki, the Japanese mathematician. This notion is originated from two different ways. One of the motivations is based on set theory. In set theory, there are three most elementary and fundamental operations introduced by L. Kantorovic and E. Livenson to make a new set from the old sets. These fundamental operations are union, intersection and the set difference. Then, as a generalization of those three operations and properties, we have the notion of Boolean algebra. If we take both of the union and the intersection, then as a general algebra, the notion of distributive lattice is obtained. Moreover, if we consider the notion of union or intersection, we have the notion of an upper semilattice or a lower semilattice. But the notion of set difference was not considered systematically before K. Iseki.

Another motivation is taken from classical and non-classical propositional calcului. There are some systems which contain the only implication functor among the logical functors. These examples are the systems of positive implicational calculus, weak positive implicational calculus by A. Church, and BCI, BCK-systems by C. AS. Meredith.

We know the following simple relations in set theory:

$$(A - B) - (A - C) \subset C - B$$

$$A - (A - B) \subset B$$

In propositional calcului, these relations are denoted by

$$(p \longrightarrow q) \longrightarrow ((q \longrightarrow r) \longrightarrow (p \longrightarrow r))$$

$$p \longrightarrow ((p \longrightarrow q) \longrightarrow q)$$

From these relationships, K. Iseki introduced a new notion called a BCK-algebra.

Definition 1.1.1[27]. Let X be a set with a binary operation " $*$ " and a constant "0". Then $(X, *, 0)$ is called a BCK-algebra if it satisfies the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $0 * x = 0$,

(V) $x * y = 0$ and $y * x = 0$ imply $x = y$.

for all $x, y, z \in X$.

For brevity we also call X a BCK-algebra. If in X we define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$, then $(X, *, 0)$ is a BCK-algebra if and only if it satisfies the following: For all $x, y \in X$;

(i) $(x * y) * (x * z) \leq z * y$,

(ii) $x * (x * y) \leq y$,

(iii) $x \leq x$,

(iv) $0 \leq x$,

(v) $x \leq y$ and $y \leq x$ imply $x = y$.

Theorem 1.1.2[27]. In a BCK-algebra $(X, *, 0)$, we have the following properties:

For all $x, y, z \in H$;

(i) $x \leq y$ implies $z * y \leq z * x$,

(ii) $x \leq y$ and $y \leq z$ implies $x \leq z$,

(iii) $(x * y) * z = (x * z) * y$,

(iv) $x * y \leq z$ implies $x * z \leq y$,

(v) $(x * z) * (y * z) \leq x * y$,

(vi) $x \leq y$ implies $x * z \leq y * z$,

(vii) $x * y \leq x$,

(vii) $x * 0 = x$.

Definition 1.1.3[27]. If there is an element $1 \in X$ of a BCK-algebra X satisfying $x \leq 1$ for all $x \in X$, then the element 1 is called the unit of X. A BCK-algebra with a unit is called to be bounded.

Theorem 1.1.4[27]. Let $(X, *, 0)$ be a BCK-algebra and $1 \notin X$. We define the operation " $*'$ " on $\overline{X} = X \cup \{1\}$ as follows

$$x *' y = \begin{cases} x * y & \text{if } x, y \in X \\ \{0\} & \text{if } x \in X \text{ and } y = 1 \\ 1 & \text{if } x = 1 \text{ and } y \in X \\ 0 & \text{if } x = y = 1, \end{cases}$$

Then $(\overline{X}, *', 0)$ is a bounded BCK-algebra with unit 1.

Definition 1.1.5[27]. Let $(X_1, *_1, 0)$ and $(X_2, *_2, 0)$ be two BCK-algebras and $X_1 \cap X_2 = \{0\}$. Suppose $X = X_1 \cup X_2$, define $*$ on X as follows:

$$x * y = \begin{cases} x *_1 y & \text{if } x \text{ and } y \text{ belong to } X_1 \\ x *_2 y & \text{if } x \text{ and } y \text{ belong to } X_2 \\ x & \text{if } x \text{ and } y \text{ do not belong to the same algebra} \end{cases}$$

Next we will verify that $(X, *, 0)$ is a BCK-algebra, this algebra is called to be the *union* of $(X_1, *_1, 0)$ and $(X_2, *_2, 0)$, denoted by $X_1 \bigoplus X_2$.

Theorem 1.1.6[27]. Let $(X_i, *_i, 0_i), (i \in I)$ be an indexed family of BCK-algebras and let $\prod\limits_{i \in I} X_i$ be the set of all mappings $f : I \longrightarrow \cup_{i \in I} X_i$, where $f(i) \in X_i$ for all $i \in I$. For $f, g \in \prod\limits_{i \in I} X_i$, we define $f * g$ by $(f * g)(i) = f(i) * g(i)$ for every $i \in I$, and by 0 we mean $0(i) = 0_i, \forall i \in I$. Then $(\prod\limits_{i \in I} X_i, *, 0)$ is a BCK-algebra, which is called the *direct product* of $X_i (i \in I)$.

Definition 1.1.7[27]. Let $(X, *, 0)$ be a BCK-algebra and S be a non-empty subset of X. Then S is called to be a *subalgebra* of X if, for any $x, y \in S, x * y \in S$, i.e., S is closed under the binary operation $*$ of X.

Definition 1.1.8[27]. A non-empty subset I of a BCK-algebra X is called an *ideal* of X if for all $x, y \in X$:

(i) $0 \in I$

(ii) $x * y \in I$ and $y \in I$ imply that $x \in I$.

Definition 1.1.9[27]. In a given BCK-algebra $(X, *, 0)$, a non-empty subset I of X is said to be a positive implicative ideal if it satisfies, for all x, y, z in X,

(i) $0 \in I$,

(ii) $(x * y) * z \in I$ and $y * z \in I$ imply $x * z \in I$.

Definition 1.1.10[27]. A BCK-algebra $(X, *, 0)$ is called to be positive implicative if it satisfies for all x, y and z in X,

$$(x * z) * (y * z) = (x * y) * z.$$

Definition 1.1.11[27]. Let $(X, *, 0)$ be a BCK-algebra $(X, *, 0)$ and $a, b \in X$. Define:

$$A(a, b) = \{x \in X : x * a \leq b\}$$

Obviously, 0, a and b are in $A(a, b)$. If for all $x, y \in X$, $A(x, y)$ has a greatest element, written $x + y$, then the BCK-algebra is called to be with *condition* (S).

Theorem 1.1.12[27]. Any positive implicative ideal must be an ideal, but the inverse is not true.

Definition 1.1.13[27]. A non-empty subset I of a BCK-algebra X is said to be a *Varlet ideal* if , for all $x, y \in X$,

(i) $x \in I$ and $y \leq x$ imply $y \in I$,

(ii) $x \in I$ and $y \in I$ imply that there exists $z \in I$ such that $x \leq z$ and $y \leq z$.

Definition 1.1.14[27]. Let X be a BCK-algebra with condition (S) and I be a non-empty subset of X. Then I is called to be an *additive ideal* if , for all $x, y \in X$,

(i) $x \in I$ and $y \leq x$ imply $y \in I$,

(ii) $x \in I$ and $y \in I$ imply $x + y \in I$.

The theory of hypercompositional structure has been introduced by F. Marty in 1934 during the 8^{th} congress of Scandinavian Mathematicians, where he presented his work [22]. Several references have also been made to H. S. Wall who presented his paper [28] in 1937. Unfortunately Marty himself did not present more than 3 or 4 papers, because he died very young during the world war II. So F. Marty introduced the notion of a hypergroup. Today the research in the field of hypercompositional structures is very vivid. In many universities in the world there are working teams on this theory.

Now in this dissertation we use the hyperstructures on BCK-algebras and define two new notions, hyper BCK-algebra and hyper$K$-algebra, which are studied in Chapters 2 and 3.

Definition 1.1.15. Let H be a non-empty set and "$\circ$" be a function from $H \times H$ to $P^*(H) = P(H) \setminus \{\emptyset\}$. Then "$\circ$" is called a hyperoperation on H.

Definition 1.1.16. For two non-empty subsets A and B of H, denote by $A \circ B$ the set

$$\bigcup_{a \in A, b \in B} a \circ b$$

.

Notation. Let "$\circ$" be a hyperopration on H and $a \in H, A \in P^*(H)$. Then by $a \circ A$ and $A \circ a$ we mean $\{a\} \circ A$ and $A \circ \{a\}$ respectively.

Chapter 2

Hyper BCK-algebras

This chapter has three sections as follows:

In section 2.1, we introduce the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra, and obtain some basic results. We also introduce the notion of a hyper BCK-ideal, weak hyper BCK-ideal, strong hyper BCK-ideal and a reflexive hyper BCK-ideal in hyper BCK-algebras, and give some relations between these notions.

In section 2.2, we consider a hyper BCK-algebra that satisfies the hypercondition, and investigate related properties. We also introduce the notion of Varlet hyper BCK-ideals and additive hyper BCK-ideals, and state some relations between these notions.

In section 2.3, first we give the notion of hyper BCK-algebras which satisfy the simple or normal conditions. After that by considering the notion of an isomorphism between two hyper BCK-algebras, we determine all of hyper BCK-algebras of order 3, which satisfy either the simple condition or the normal condition. In fact there are 19 hyper BCK-algebras of order 3 up to isomorphism.

2.1 Hyper BCK-algebras

Definition 2.1.1 By a *hyper BCK-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0, satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$, (HK3) $x \circ H \ll \{x\}$,

(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

Example 2.1.2. (1) Let $(H, *, 0)$ be a BCK-algebra and define a hyper operation "$\circ$" on H by $x \circ y = \{x * y\}$ for all $x, y \in H$. Then $(H, \circ, 0)$ is a hyper BCK-algebra.

(2) Define a hyper operation "$\circ$" on $H := [0, \infty)$ as a subset of real numbers by

$$x \circ y := \begin{cases} [0, x] & \text{if } x \leq y \\ (0, y] & \text{if } x > y \neq 0 \\ \{x\} & \text{if } y = 0 \end{cases}$$

for all $x, y \in H$. Then $(H, \circ, 0)$ is a hyper BCK-algebra.

(3) Let $H = \{0, 1, 2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Then $(H, \circ, 0)$ is a hyper BCK-algebra

Proposition 2.1.3. In a hyper BCK-algebra $(H, \circ, 0)$, the condition (HK3) is equivalent to the condition:

(i) $x \circ y \ll \{x\}$ for all $x, y \in H$.