٣٩٤٠٨

IN THE NAME OF ALLAH

RINGS WHOSE CYCLICS ARE ESSENTIALLY EMBEDDABEL IN PROJECTIVE MODULES

BY

JACOB AHMADY

THESIS

SUBMITTED TO THE SCHOOL OF GRADUATE STUDIES IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF *MASTER OF SCIENCES (M.Sc.)*

IN

PURE MATHEMATICS

SHIRAZ UNIVERSITY

SHIRAZ, IRAN

EVALUATED AND APPROVED BY THE THESIS COMMITTEE AS:

M. Ershad .... ERSHAD, M., Ph.D., ASSISTANT. PROF. OF MATH. (CHAIRMAN)

H. Sharif .... SHARIF, H., Ph.D., PROF. OF MATH.

.... HAKIM HASHEMI, M., Ph.D., ASSISTANT. PROF. OF MATH.

SEPTEMBER 2000

TO:

MY PARENTS

AND

TO ALL THOSE WHOM I LOVE

ACKNOWLEDGMENT

Now that another page is turned, because of acknowledgement to God, I am gratefull to the suffering of all those that I am their guideness's own. I appreciate from my dear and honourable professor Mr. Dr. Majid Ershad due to his unsparingly guidnesses and efforts. Also, I appreciate from honourable professors, Mr. Dr. H. Sharif and Dr. Mehdi Hakim Hashemi, who undertook the consultant of this smallest student. At the end, I introduce my endlessness grateful to all of my teachers.

ABSTRACT

RINGS WHOSE CYCLICS ARE ESSENTIALLY EMBEDDABLE IN PROJECTIVE MODULES

BY

M. AHMADY

For the first time Nakayama introduced QF-ring. In 1967 Carl.Faith and Elbert A. Walker showed that R is QF-ring if and only if each injective right R-module is projective if and only if each injective left R-modules is projective.

In 1987 S.K.Jain and S.R.Lopez-Permouth proved that every ring homomorphic images of R has the property that each cyclic S-module is essentialy embeddable in direct summand of S if and only if R is a direct sum of right uniserial rings if and only if R is a semiperfect ring whose cyclics are essentially embeddable in a direct summand of R.

TABLE OF CONTENTS

CHAPTER I
INTRODUCTION

Throughout our dissertation, unless stated otherwise, all modules are unital.

As usual mod-R (R-mod) denotes the category of right(left) R-modules.

1.1 Categories

Definition 1.1.1 A category is a class $\mathcal{C}$ of objects (denoted by $A, B, C, \ldots$) together with (i) a class of disjoint sets, denoted by hom (A, B), one for each pair of objects in $\mathcal{C}$; (an element f of hom (A, B) is called a morphism from A to B and is denoted by $f : A \longrightarrow B$) such that (ii) for each triple (A, B, C) of objects of $\mathcal{C}$ a function *hom* $(B, C)\times$ *hom* $(A, B) \longrightarrow$ *hom*(A, C) (for morphisms $f : A \longrightarrow B, g : B \longrightarrow C$ this function is written by $(g, f) \longrightarrow g \circ f$ and $g \circ f : A \longrightarrow C$ is called the composite of f and g); all subject to the axioms: (1) Associativity: If $f : A \longrightarrow B, g : B \longrightarrow C, h : C \longrightarrow D$ are morphisms of $\mathcal{C}$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(2) Identity: for each object B of $\mathcal{C}$ there exists a morphism $1_B : B \longrightarrow B$ such that for any $f : A \longrightarrow B, g : B \longrightarrow C, 1_B \circ f = f$ and $g \circ 1_B = g$.

1.2 Indecomposable module

Definition 1.2.1. A non-zero module M is indecomposable if o and

M are its only direct summands.

A pair of idempotents e_1 and e_2 in a ring R are said to be orthogonal if $e_1 e_2 = 0 = e_2 e_1$.

An idempotent $e \in R$ is called a primitive idempotent in case $e \neq 0$ and for every pair e_1, e_2 of orthogonal idempotents

$$e = e_1 + e_2 \text{ implies } e_1 = 0 \text{ or } e_2 = 0.$$

If $e = e^2 \in R$, then e and $1 - e$ are orthogonal idempotents such that $1 = e + (1 - e)$.

Proposition 1.2.1. Let M be a non-zero module. Then the following are equivalent:

(a) M is indecomposable.

(b) 0 and 1 are the only idempotents in $End(M)$.

(c) 1 is a primitive idempotent in $End(M)$.

Proof. See [1, 5.10].

Proposition 1.2.2. Let e be a non-zero idempotent endemorphism of a left module M. Then the direct summand Me of M is indecomposable if and only if e is a primitive idempotent in $End(M)$.

Proof. See [1, 5.11].

Proposition 1.2.3. Let $e \in R$ be a non-zero idempotent. Then the following statements are equivalent:

(a) e is a primitive idempotent;

(b) Re is a primitive left ideal of R;

(c) eR is a primitive right ideal of R;

(d) Re is an idecomposable direct summand of ${}_RR$;

(e) eR is an indecomposable direct summand of R_R;

(f) The ring eRe has exactly one non-zero idempotent, namely e.

Proof. See [1, 7.4].

1.3 Free Modules

Definition 1.2.2. A subset X of a left R-module M is said to be linearly indepenedent provided for distinct $x_1, \dots, x_n \in X$ and $r_i \in R, r_1x_1 + \cdots + r_nx_n = 0$ implies that $r_i = 0$ for every i. If M is generatored as an R-module by a set Y then we say that Y spans M. If R has an identity and M is unitary, then Y spans M if and only if every element of M can be written as a linear combination $r_1y_1 + r_2y_2 + \cdots + r_ny_n (r_i \in R, y_i \in Y)$. A linearly independent subset of M that spans M is called a basis of M.

Theorem 1.2.4. Let R be a ring with identity, the following conditions on a unitary left R-module F are equivalent:

(i) F has a nonempty basis.

(ii) F is the internal direct sum of a family of cyclic modules, each of which is isomorphic as left R-module to R.

(iii) F is R-module isomorphic to a direct sum of copies of the left R-module R.

(iv) There exists a nonempty set X and function $i : X \longrightarrow F$ with the following property:

Given any unitary R-module M and function $F : X \longrightarrow M$, there is a unique R-module homomorphism $\bar{f} : F \longrightarrow M$ such that $\bar{f}i = f$. In other words, F is a free object in the category of unitary R-modules.

Proof. See [8, iv.2.1].

Definition 1.2.3. A unitary module F over a ring R with identity which satisfies the equivalent conditions of the above theorem, is called a free R-module.

1.4 Projective and injective modules

Definition 1.2.4. Let R be a ring. A right R-module P is called projective if given any diagram of R-module homomorphisms

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow h & & \\ A & \xrightarrow{g} & B & \longrightarrow & 0 \end{array}$$

with A and B are right R-modules and the bottom row is exact (that is g is an epimorphism), there exists an R-module homomorphism f:$P \longrightarrow A$ such that the diagram

$$\begin{array}{ccccc} & & P & & \\ & f \swarrow & \downarrow h & & \\ A & \xrightarrow{g} & B & \longrightarrow & 0 \end{array}$$

is commutative (that is $gf = h$).

Proposition 1.2.5. Every free right module F over a ring R with identity is projective.

Proof. See [8, 3.2].

Theorem 1.2.6. Let R be a ring. The following conditions on a right R-module P are equivalent:

(1) P is projective.

(2) If P is a factor module of any module M, then P is a direct factor of M.

(3) P is direct summand of a free module.

Proof. See [8, 3.4].

Definition 1.2.5. A right R-module E is called injective, if given any diagram of R-module homomorphism

$$\begin{array}{ccccc} & & E & & \\ & & f\uparrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B \\ & & & g & \end{array}$$

which A and B are right R-modules and the top row is exact (that is g is a monomorphism), there exists an R-module homomorphism $h : B \longrightarrow E$ such that the diagram

$$\begin{array}{ccccc} & & E & & \\ & & f\uparrow & \nwarrow h & \\ 0 & \longrightarrow & A & \longrightarrow & B \\ & & & g & \end{array}$$

is commutative (that is $hg = f$).

Proposition 1.2.7. A direct product of R-modules $\Pi_{i\in I}J_i$ is injective

Dually, a submodule K of M is superfluous (or small) in M, abbreviated $K << M$, in case for every submodule $L \leq M$, $K + L = M$ implies $L = M$.

A monomorphism $f : K \longrightarrow M$ is said to be essential in case $Imf \trianglelefteq M$. An epimorphism $g : M \longrightarrow N$ is superfluous in case $kerg << M$.

Theorem 1.6.1 Let M be a module with submodules $K \leq N \leq M$ and $H \leq M$. Then

(1) $K \trianglelefteq M$ iff $K \trianglelefteq N$ and $N \trianglelefteq M$;

(2) $H \cap K \trianglelefteq M$ iff $H \trianglelefteq M$ and $K \trianglelefteq M$.

Proof. (1) LEt $K \trianglelefteq M$ and suppose $0 \neq L \leq M$, then $L \cap K \neq 0$. In particular this is true if $L \leq N$, so $K \trianglelefteq N$. But also $K \leq N$ so $L \cap N \neq 0$ whence $N \trianglelefteq M$.

Conversely, if $K \trianglelefteq N$ and $N \trianglelefteq M$ and $L \leq M$, then $L \cap K = 0$ implies $L \cap N = 0$ implies $L = 0$.

(2) one implication follows at once from (1). For the other, suppose $H \trianglelefteq M$ and $K \trianglelefteq M$ with $L \cap H \cap K = 0$, then $L \cap H = 0$; because $K \trianglelefteq M$. whence $L = 0$ because $H \trianglelefteq M$.

Theorem 1.6.2. Let M be a module with submodules $K \leq N \leq M$ and $H \leq M$. Then

(1) $N << M$ iff $K << M$ and $\frac{N}{K} << \frac{M}{K}$;

(2) $H + K << M$ iff $H << M$ and $K << M$.

Proof. See [1, 5.17].

Theorem 1.6.3. Suppose that $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, and $M = M_1 \oplus M_2$; then if and only if J_i is injective for every $i \in I$.

Proof. See [8, IV.3.7].

Proposition 1.2.8. Every unitary module A over a ring R with identity may be embedded in an injective R-module.

Proof. See[8, 3.12].

1.5 Idempotents

Definition 1.2.5. Let R be a ring. An element $e \in R$ is an idempotent in case $e^2 = e$. A ring always has at least two idempotents, namely 0 and 1. An idempotent e of R is a central idempotent in case it is in the center of R.

Definition 1.2.6. Let I be an ideal in a ring R and let $g + I$ be an idempotent element of $\frac{R}{I}$. We say that this idempotent can be lifted (to e) modulo I in case there is an idempotent $e \in R$ such that $g+I = e+I$. We say that idempotents lift modulo I in case every idempotent in $\frac{R}{I}$ can be lifted to an idempotent in R.

A finite orthogonal set of idempotents $e_1, \ldots, e_n$ in a ring R is said to be complete in case $e_1 + \cdots + e_n = 1 \in R$.

1.6 Module

Definition 1.2.7. A submodule K of M is essential (or large) in M, abbreviated $K \trianglelefteq M$, in case for every submodule $L \leq M$, $K \cap L = 0$ implies $L = 0$.

Dually, a submodule K of M is superfluous (or small) in M, abbreviated $K << M$, in case for every submodule $L \leq M$, $K + L = M$ implies $L = M$.

A monomorphism $f : K \longrightarrow M$ is said to be essential in case $Imf \trianglelefteq M$. An epimorphism $g : M \longrightarrow N$ is superfluous in case $kerg << M$.

Theorem 1.6.1 Let M be a module with submodules $K \leq N \leq M$ and $H \leq M$. Then

(1) $K \trianglelefteq M$ iff $K \trianglelefteq N$ and $N \trianglelefteq M$;

(2) $H \cap K \trianglelefteq M$ iff $H \trianglelefteq M$ and $K \trianglelefteq M$.

Proof. (1) LEt $K \trianglelefteq M$ and suppose $0 \neq L \leq M$, then $L \cap K \neq 0$. In particular this is true if $L \leq N$, so $K \trianglelefteq N$. But also $K \leq N$ so $L \cap N \neq 0$ whence $N \trianglelefteq M$.

Conversely, if $K \trianglelefteq N$ and $N \trianglelefteq M$ and $L \leq M$, then $L \cap K = 0$ implies $L \cap N = 0$ implies $L = 0$.

(2) one implication follows at once from (1). For the other, suppose $H \trianglelefteq M$ and $K \trianglelefteq M$ with $L \cap H \cap K = 0$, then $L \cap H = 0$; because $K \trianglelefteq M$. whence $L = 0$ because $H \trianglelefteq M$.

Theorem 1.6.2. Let M be a module with submodules $K \leq N \leq M$ and $H \leq M$. Then

(1) $N << M$ iff $K << M$ and $\frac{N}{K} << \frac{M}{K}$;

(2) $H + K << M$ iff $H << M$ and $K << M$.

Proof. See [1, 5.17].

Theorem 1.6.3. Suppose that $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, and $M = M_1 \oplus M_2$; then

(1) $K_1 \oplus K_2 << M_1 \oplus M_2$ iff $K_1 << M_1$ and $K_2 << M_2$;

(2) $K_1 \oplus K_2 \trianglelefteq M_1 \oplus M_2$ iff $K_1 \trianglelefteq M_1$ and $K_2 \trianglelefteq M_2$.

Proof. See [1, 5.20].

Definition 1.2.9. A nonzero module H is uniform in case each of its non-zero submodules is essential in H.

Definition 1.2.10. Let $(T_\alpha)_{\alpha \in A}$ be an indexed set of simple submodules of M. If M is the direct sum of this set, then $M = \oplus_A T_\alpha$ is a semisimple decomposition of M. A module M is said to be semisimple in case it has a semisimple decomposition. Clearly every simple module is semisimple.

Definition 1.2.11. The ring R is called left semisimple when the left R-module R is semisimple. Similarly we define a right semisimple ring.

Theorem 1.2.12. For a left R-module the following statements are equivalent:

(a) M is semisimple;

(b) M is generated by simple modules;

(c) M is the sum of some set of simple submodules;

(d) M is the sum of its simple submodules;

(e) Every submodule of M is a direct summand.

Proof. See [1,9.6].

Definition 1.2.12. A commutative ring is a local ring in case it has a unique maximal ideal.

Definition 1.3.1. A ring R is left(right) self-injective in case ${}_RR(R_R)$ is injective.

Definition 1.3.2. An injective hull (or injective envelope) ffor a module A is any injective module which is an essential extension of A.

Theorem 1.3.2. In the category of left R-modules over a ring R:

(1) M is injective if and only if $M = E(M)$;

(2) If $M \trianglelefteq N$, then $E(M) = E(N)$;

(3) If $M \leq Q$, with Q injective, then $Q = E(M) \oplus E'$;

(4) If $\oplus_A E(M_\alpha)$ is injective (for instance, if A is finite) then

$$E(Q_A M_\alpha) = \oplus_A E(M_\alpha).$$

Proof. See [1, 18.12].

1.3 Composition series

Definition 1.3.1. Let M be a non-zero module. A finite chain of $n+1$ submodules of M $M = M_0 > M_1 > \cdots > M_n = 0$ is called a composition series of length n for M provided that $\frac{M_{i-1}}{M_i}$ is simple $(i = 1, 2, \ldots, n)$; *i.e*, provided that each term in the chain is maximal in its predecessor.

Let M be an arbitary module and let $L \leq M$. Then whether or not L is a term in a composition series for M, if L has a maximal submodule