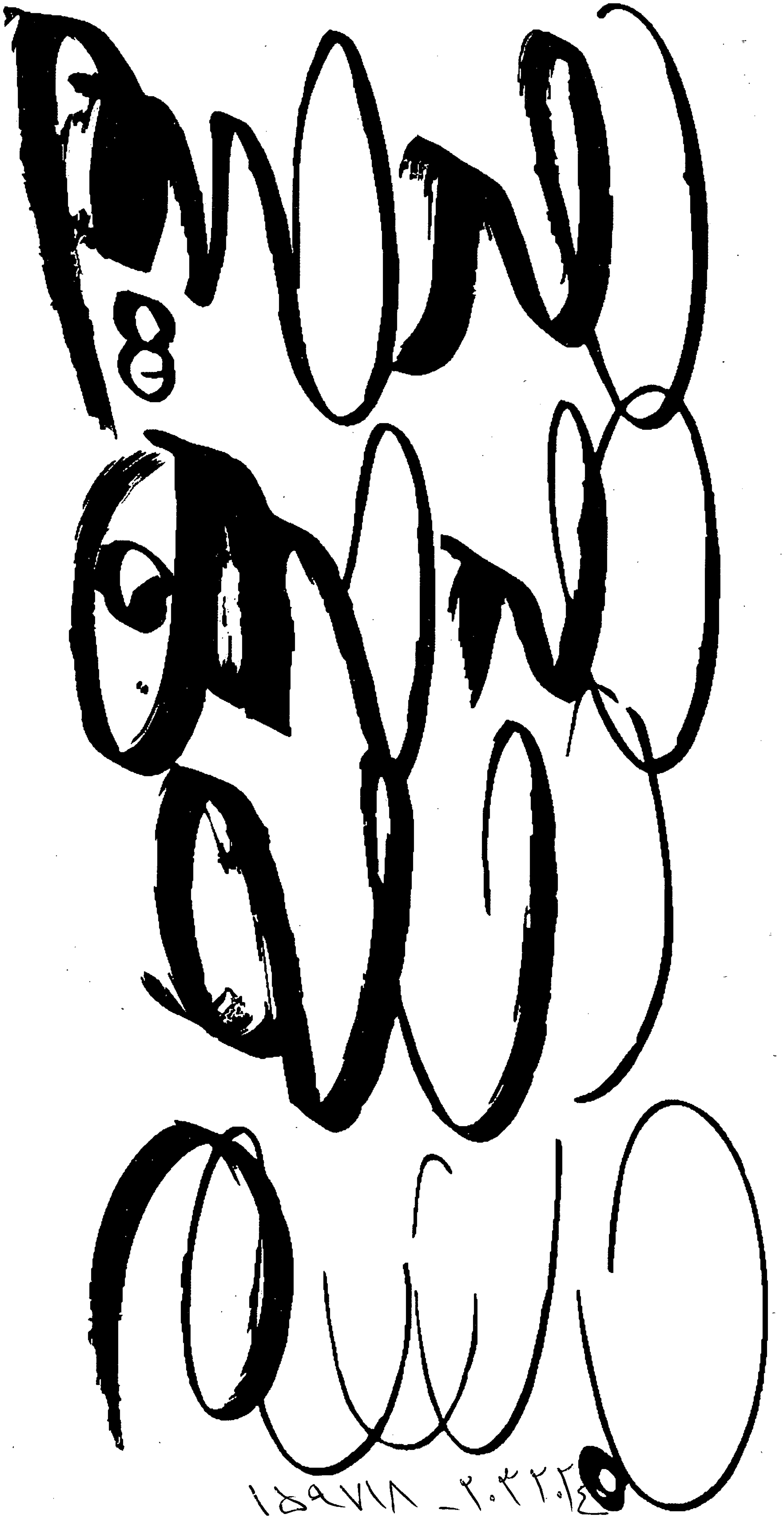

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APPLICATIONS OF WAVELETS AND FRAMES IN ADAPTIVE SOLUTIONS TO OPERATOR EQUATIONS

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The undersigned hereby certify that they have read and recommend to the Faculty of Mathematics Statistics and Computer for acceptance a thesis entitled "**Applications of wavelets and frames in adaptive solutions to operator equations** " by **Hassan Jamali** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

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Dedicated

My kind wife

and

my beloved dougters

Sahel and Baran

Table of Contents

Abstract

The aim of this work is to construct some iterative algorithms in order to present an adaptive solution of the operator equation $\mathrm{L}u = f$ where, $\mathrm{L} : H \longrightarrow H^*$, H is a separable Hilbert space with dual H^* and $f \in H^*$. As typical example we think of linear differential or integral equations in variational form. By using wavelets or frames we transform the problem to an equivalent ℓ_2-problem and then construct the adaptive algorithms for solving the problem. Then we show how these algorithms displays optimal approximation and complexity properties.

The contexts of this thesis are presented in four chapters. First, we will review the concepts of wavelet, frame, and N-term approximation to obtain the equivalent problems.

In the second chapter, we use frames to construct corresponding trial spaces for an adaptive Galerkin scheme and design an algorithm in order to give an adaptive approximation solution to the problem.

In the third chapter, by using a wavelet frame we construct an adaptive algorithm for solving the problem.

Finally, as an example we will present an adaptive wavelet scheme to solve the generalized Stokes problem. Using divergence free wavelets, the problem is transformed into an equivalent matrix vector system, that leads to a positive definite system of reduced size for the velocity. Then we prove that this adaptive method has optimal computational complexity that is it recovers an approximate solution with desired accuracy at a computational expense that stays proportional to the number of terms in a corresponding wavelet-best N-term approximation.

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INTRODUCTION

Let H be a separable Hilbert space with dual H^*. We consider the problem of finding $u \in H$ such that

$$\text{L}u = f, \tag{0.0.1}$$

where $\text{L} : H \to H^*$ is a symmetric, positive definite and bounded invertible linear operator. As typical example we think of linear differential or integral equations in variational form. Assuming that we have a Riesz basis Ψ for H available, which we formally view as a column vector, by writing $u = U^t\Psi$, the above problem is equivalent to find $U \in \ell_2$ satisfying the infinite matrix-vector system

$$GU = \mathbf{f}, \tag{0.0.2}$$

where $G = \langle \Psi, \text{L}\Psi \rangle : \ell_2 \to \ell_2$ is bounded invertible, and $\mathbf{f} = \langle \Psi, f \rangle \in \ell_2$. Here $\langle , \rangle$ denotes the duality product on (H, H^*). Since L is bounded and invertible, (0.0.1) has a unique solution and

$$\|\text{L}u\|_{H^*} \simeq \|u\|_H, \quad u \in H,$$

that is there are two positive constants d_1, d_2 such that $d_1\|\text{L}u\|_{H^*} \leq \|u\|_H \leq d_2\|\text{L}u\|_{H^*}$ for all $u \in H$. Also the bilinear form a defined by

$$a(u, v) := \langle \text{L}u, v \rangle$$

is symmetric, positive definite and *elliptic* in the sense that

$$a(v, v) \simeq \|v\|_H^2. \tag{0.0.3}$$

It follows that H is a Hilbert space with respect to the inner product a with an equivalent energy norm $\|.\|_a^2 := a(.,.)$. In [14, 15], an iterative adaptive method for solving this system has been developed that for a given tolerance $\epsilon > 0$ yields an approximate solution U_ϵ, where the number of operations and storage locations it requires is of the same order as the length of the smallest N-term approximation for U on distance ϵ. This means that the method has optimal computational complexity. When L and G are symmetric and positive definite, the method consists of the application of the simple damped Richardson iteration onto the infinite system, where the multiplication of G with current, finitely supported approximation vector for U is replaced by an adaptive approximation. The analysis of adaptive numerical scheme for operator equations is a field of enormous current interest. Recent developments for instance in the finite element context, indeed indicate their promising potential [4, 5, 6, 10, 29, 32]. Moreover, it has also turned out that adaptive schemes based on wavelets have several important advantages. A typical algorithm uses information gained during a given stage of the computation to produce a new mesh for the next iteration. Thus, the adaptive procedure depends on the current numerical resolution of u. The wavelet methodology differs from other conventional schemes in so far as direct use of bases is made which span appropriate complements between successive approximation spaces. The original problem is first transformed with the aid of suitable wavelet bases into an equivalent problem that is well posed in Euclidian metric. Then one seeks an iteration scheme for the infinite dimensional problem for which the error is reduced at each step by at least a fixed ratio. Then, at last, the application of the involved (infinite dimensional) operators is carried out approximately in an adaptive way within suitable dynamically updated accuracy tolerances. First natural steps were to use multiresolution spaces spanned by wavelets (or correspondingly scaling functions) as test and trial spaces for Galerkin methods. In connection with elliptic boundary value problems suitable wavelet bases lead to asymptotically optimal preconditioners in terms of simple diagonal scalings in wavelet coordinates. Asymptotically optimal means here that the resulting linear systems can be solved within discretization error accuracy at a computational expense that stays proportional to the problem size. In that sense such schemes are, in principle, comparable with multigrid methods. In the wavelet context it is natural to derive information from the size of the wavelet coefficients of current approximations, see [3, 7, 9, 17, 18, 20].

Usually, the operator under consideration is defined on a bounded domain $\Omega \subset \mathbb{R}^d$ or on a closed manifold. Therefore the construction of a wavelet basis with specific properties on this domain or on the manifold is needed. Although there exist by now several constructive methods such as [22, 23], none of them seems to be fully satisfying in the sense that some serious drawbacks such as stability problems cannot be avoided. One way out could be to use a fictitious domain method [42], however, then the compressibility of the problem might be reduced. Motivated by these difficulties, we therefore suggest to use a slightly weaker concept, namely frames.

Frames were first introduced by Duffin and Schaffer [30] in the context of nonharmonic Furier series. . Outside of signal processing, frames did not seem to generate much interest until the ground breaking work of Daubechies, Grossmann, and Meyer

[26] in 1986. Since then the theory of frames began to be more widely studied. Traditionally, frames have been used in signal processing, image processing, data compression, and sampling theory. Recently, the theory is beginning to grow even more rapidly, since several new applications have been developed. For example, frames are now used to migrate the effect of losses in packet-based communication systems and hence to improve the robustness of data transmission [36] and to design high-rate constellations with full diversity in multiple-antenna code design [37].

Every element of the Hilbert space H has an expansion with respect to the frame elements, but in contrast to stable multiscale bases, its representation is not necessarily unique. Therefore frame expansions may contain some redundancy. The redundancy of a frame provide to play an important role in practical problems where stability and error tolerance are fundamental as, for example, denoising, pattern matching, or irregular sampling problems [33]. Moreover since one is working with a weaker concept, the concrete construction of a frame is usually much simpler when compared to stable multiscale bases. The potential of frames in numerical analysis is an almost unexplored field. One of the first interesting attempts to use frames for numerical simulation is [43], being a pioneering approach to the application of wavelet frames to the adaptive solution of operator equations.

In the adaptive method for solving (0.0.1), the use of a frame instead of a Riesz basis gives also rise to a problem. Since in the adaptive method the matrix-vector product is replaced by an adaptive approximation, each time it is invoked it gives an error that might have a component in the, nontrivial, kernel of G. Also clean-up or coarsening step may introduce such components. Just because these components are in the kernel of G , they will not be affected by subsequent Richardson steps, meaning that in the cause of the iteration the component of the current approximation in the kernel of G may increase. Although this component has no influence on the obtained approximation for the solution of (0.0.1), that is, after forming the series with frame elements, it might be responsible costs of each iteration. Under some technical assumption on the frame, specially on the projector onto the complement of the kernel of G in ℓ_2, it is proved that the above effect will not occur or only to such an extent that also in the frame case the adaptive method has optimal computational complexity [43].

Chapter 1

BASIC CONCEPTS

We investigate wavelets and frames to obtain the equivalent problem of finding solution U to equation (0.0.2). Also we will review the concept of N-term approximation in order to understanding the properties of U that determine its approximability.

1.1 Scaling Functions and Multiresolution Analysis

In this section, we will briefly review the basic concepts of wavelets and multiresolution analysis for the construction of wavelets. In general, a set of functions $\{\psi_i\}_{i=1,...,N}$ is called a system of (mother) wavelets, if the scaled and translated versions of these functions form a basis of $L^2(\mathbb{R}^n)$. This means that the resulting L^2-basis is obtained from very easily implementable algebraic modifications of a finite number of these functions. Let $\mathbb{E} := \{(e_1, ..., e_n)^t : e_i \in \{0, 1\}\}$ and $\mathbb{E}^* := \mathbb{E}\backslash\{0\}$, a set of functions $\{\psi_e\}_{e\in\mathbb{E}^*} \subset L^2(\mathbb{R}^n)$ is called a system of (mother) wavelets, provided the set

$$\{\psi_{e,j,k}(.) := 2^{nj/2}\psi_e(2^j. - k) : e \in \mathbb{E}^*, j \in \mathbb{Z}^n\}, \tag{1.1.1}$$

form a basis of $L^2(\mathbb{R}^n)$. The most powerful tool for the construction of wavelets is the multiresolution approximation of functions, which introduced by Mallat and

Meyer [40, 41]. A *multiresolution analysis* of $L^2(\mathbb{R}^n)$ is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces $V_j \subset L^2(\mathbb{R}^n)$, which are nested, i.e., $V_j \subset V_{j+1}$ and *shift-invariant*, i.e., $f(.) \in V_j \Leftrightarrow f(. - k) \in V_j, (k \in \mathbb{Z}^n, j \in \mathbb{Z})$. Moreover the union of all V_j is dense in $L^2(\mathbb{R}^n)$, while their intersection is $\{0\}$. Finally, one requires $f(.) \in V_j$ if and only if $f(2.) \in V_{j+1}$ for all $j \in \mathbb{Z}$ and that a function $\phi \in V_0$ exists such that V_0 is spanned by its integer translates, which form a Riesz-basis of V_0. This means there are positive constants A and B such that

$$A\|\xi\|_{\ell_2} \leq \| \sum_{k\in\mathbb{Z}^n} \xi_k \phi(. - k)\|_{L^2} \leq B\|\xi\|_{\ell_2}, \tag{1.1.2}$$

for all $\xi \in \ell_2(\mathbb{Z}^n)$. Such a function is called *generator* of the multiresolution analysis. Thus

$$V_j = \overline{span} \ \ \{\phi_{j,k} : k \in \mathbb{Z}^n\}$$

and hence the generator ϕ is a *refinable function* (then ϕ is called to be *a-refinable*), i.e.,

$$\phi(x) = \sum_{k\in\mathbb{Z}^n} a_k \phi(2x - k). \tag{1.1.3}$$

The Laurent-series

$$a(z) := \sum_{k\in\mathbb{Z}^n} a_k z^k \tag{1.1.4}$$

is called the *symbol* of ϕ, where $z = e^{-i\xi}$ and $\xi \in \mathbb{R}^n$ (thus z lies in the n-dimensional torus $\mathbb{T}^n := \{z \in \mathbb{C}^n : |z_i| = 1, 1 \leq i \leq n\}$). Because of the approximating property $j \in \mathbb{Z}$ is often called *refinement level*.

To construct wavelets one has to find functions $\psi_e, \ \ e \in \mathbb{E}^*$, in some appropriate complement W_0 of V_0 in V_1, such that W_0 is spanned by the integer translates of ψ_e. Hence defining

$$W_j := \{f \in L^2(\mathbb{R}^n) : f(2^j.) \in W_0\},$$

one has

$$V_{j+1} = V_j \oplus W_j \quad \text{and} \quad L^2(\mathbb{R}^n) = \bigoplus_{j \in \mathbb{Z}} W_j.$$

The natural choice is to talk W_0 as the orthogonal complement of V_0 in V_1. In this case one searches for functions ψ_e such that $\{\psi_{e,0,k} : e \in \mathbb{E}^*, k \in \mathbb{Z}^n\}$ forms an orthonormal basis of W_0. Hence the functions $\{\psi_{e,j,k} : e \in \mathbb{E}^*, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ form an orthonormal basis of $L^2(\mathbb{R}^n)$. In particular, we are interested in wavelets with compact support. Daubechies [25] was the first who construct compactly supported orthonormal wavelet-bases in one dimension.

In the framework of *pre-wavelets*, one only demands that $\{\psi_{e,0,k} : e \in \mathbb{E}^*, k \in \mathbb{Z}^n\}$ forms an ℓ_2-*stable system*, i.e.,

$$\| \sum_{e \in \mathbb{E}^*} \sum_{k \in \mathbb{Z}^n} \xi_k^e \psi_e(. - k) \|_{L^2} \geq c. \sum_{e \in \mathbb{E}^*} \| \xi^e \|_{\ell_2}, \quad c > 0, \tag{1.1.5}$$

instead of requiring that the above system forms an orthonormal basis of W_0. That means we only have orthogonality between different refinement levels. The theory of pre-wavelets was described by Jia and Micchelli [38] in very general terms. This concept allows more flexibility in the construction of the functions ψ_e. Another very important property of pre-wavelets is that if one starts with a compactly supported generator ϕ, then the pre-wavelets can be chosen as compactly supported functions, too. In the orthonormal setting this is generally not true.

1.2 Frames

Assume that H is a separable Hilbert space with dual H^*, Λ is a countable set of indices and $\Psi = (\psi_\lambda)_{\lambda \in \Lambda} \subset H$ is a *frame* for H. This means that there exist constants

$0 < A \leq B < \infty$ such that

$$A\|f\|_H^2 \leq \sum_\lambda |\langle f, \psi_\lambda \rangle|^2 \leq B\|f\|_H^2, \quad \forall f \in H, \tag{1.2.1}$$

or equivalently (by the Riesz mapping),

$$A\|f\|_{H^*}^2 \leq \|f(\Psi)\|_{\ell_2}^2 \leq B\|f\|_{H^*}^2, \quad \forall f \in H^*, \tag{1.2.2}$$

where $f(\Psi) = (f(\psi_\lambda))_\lambda = (\langle f, \psi_\lambda \rangle)_\lambda$. For an index set $\widetilde{\Lambda} \subset \Lambda$, $(\psi_\lambda)_{\lambda \in \widetilde{\Lambda}}$ is called a *frame sequence*, if it is a frame for its closed span. We assume that $(\psi_\lambda)_{\lambda \in \widetilde{\Lambda}}$ is a frame sequence with bounds A, B, for all finite index sets $\widetilde{\Lambda} \subset \Lambda$.

For the frame Ψ, let $T : \ell_2(\Lambda) \to H$ be the *synthesis operator*

$$T((c_\lambda)_\lambda) = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda,$$

and let $T^* : H \to \ell_2(\Lambda)$ (or $T^* : H^* \to \ell_2(\Lambda)$) be the *analysis operator*

$$T^*(f) = (\langle f, \psi_\lambda \rangle)_\lambda.$$

Also, let $S := TT^* : H \to H$ be the *frame operator*

$$S(f) = \sum_\lambda \langle f, \psi_\lambda \rangle \psi_\lambda.$$

Note that T is surjective, T^* is injective and T^* is the adjoint of T. Because of (1.2.1) or (1.2.2) T is bounded, in fact we have

$$\|T\| = \|T^*\| \leq \sqrt{B}. \tag{1.2.3}$$

It was shown in [12], S is a positive invertible operator satisfying $AI_H \leq S \leq BI_H$ and $B^{-1}I_H \leq S^{-1} \leq A^{-1}I_H$. Also, the sequence

$$\widetilde{\Psi} = (\widetilde{\psi}_\lambda)_{\lambda \in \Lambda} = (S^{-1}\psi_\lambda)_{\lambda \in \Lambda}$$

is a frame (called the *canonical dual frame*) for H with bounds B^{-1}, A^{-1}. Every $f \in H$ has the expansion

$$f = \sum_{\lambda} \langle f, \widetilde{\psi}_\lambda \rangle \psi_\lambda = \sum_{\lambda} \langle f, \psi_\lambda \rangle \widetilde{\psi}_\lambda.$$

Since $\mathrm{Ker}(T) = (\mathrm{Ran}(T^*))^{\perp}$, we have $\ell_2(\Lambda) = \mathrm{Ran}(T^*) \oplus \mathrm{Ker}(T)$. Thus the orthogonal projection Q of a sequence $(c_\lambda)_{\lambda \in \Lambda} \in \ell_2(\Lambda)$ onto the $\mathrm{Ran}(T^*)$ is given by

$$Q(c_\lambda)_{\lambda \in \Lambda} = (\langle \sum_{\lambda} c_\lambda S^{-1} \psi_\lambda, \psi_j \rangle)_{j \in \Lambda},$$

that is, $Q = T^* S^{-1} T : \ell_2(\Lambda) \to \ell_2(\Lambda)$. For more details see [12].

A complete sequence $(\psi_\lambda)_{\lambda \in \Lambda}$ in H is called a Riesz basis if there exist constants $0 < A, \ \ B < \infty$ such that

$$A \|C\|^2_{\ell_2(\Lambda)} \leq \| \sum_{\lambda} c_\lambda \psi_\lambda \|^2_H \leq B \|C\|^2_{\ell_2(\Lambda)}$$

for all finite sequences $C = (c_\lambda)_{\lambda \in \Lambda}$. The following conditions are equivalent:

(i) $(\psi_\lambda)_{\lambda \in \Lambda}$ is a Riesz basis for H.

(ii) The coefficients c_λ for the series expansion with ψ_λ are unique, so the synthesis operator T is injective.

(iii) The analysis operator T^* is surjective.

(iv) $(\psi_\lambda)_{\lambda \in \Lambda}$ and $(\widetilde{\psi}_\lambda)_{\lambda \in \Lambda}$ are biorthogonal.

For the proof see [12].

Also, we recall that for $a > 1, b > 0$ and $\psi \in H$, a frame for H of the form $\{a^{\frac{j}{2}} \psi(a^j x - kb)\}_{j,k \in Z}$ is called a *wavelet frame* for H (in this case H is considered as a space of functions).