In the Name of God

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Extension of Standard Uniform Algebras and Lipschitz Algebras

A Thesis Submitted in Partial Fulfilment of the Requirements for the Degree of Doctor of Philosophy (PhD)

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Dedicated to: My parents and my wife

Table of Contents

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Declaration

Unless otherwise stated, all results of Chapters 2, 3 and 4 in this PhD thesis are original.

Abstract

Let $C(X)$ denote the uniform algebra of continuous complex valued functions on a compact Hausdorff space X with the uniform norm $\|f\|_X = sup\{|f(x)| : x \in X\}$. A subalgebra A of $C(X)$, which contains the constants and separates the points of X, is called a function algebra on X. If the function algebra A on X is equipped with an algebra norm and it is complete under this norm then A is called a Banach function algebra on X. In particular, if the norm of the Banach function algebra A on X turns out to be the uniform norm then it is called a uniform algebra on X.

Let X be a compact subset of $\mathbb{C}^n$. The subalgebra of $C(X)$, which is generated by the polynomials, or rational functions on X with poles off X, is denoted by $P(X)$ or $R(X)$, respectively. The subalgebra of $C(X)$ consisting of all analytic functions on the interior of X is denoted by $A(X)$. The above subalgebras of $C(X)$ are called standard uniform algebras on X. The maximal ideal space and the Shilov boundary of the above uniform algebras as well as the approximation problem among these algebras have been studied in the last decades. For example, several criteria such as Hartogs-Rosenthal Theorem and Vitushkin's Theorem for the equality $R(X) = C(X)$ have been presented. Moreover, the maximal ideal spaces of $P(X)$, $R(X)$ and $A(X)$ have been characterized.

We extend the above standard uniform algebras as follows: Let X and K be compact subsets of $\mathbb{C}^n$ and $K \subseteq X$. We take $P(X, K) = \{f \in C(X) : f|_K \in P(K)\}$, $R(X, K) = \{f \in C(X) : f|_K \in R(K)\}$, $A(X, K) = \{f \in C(X) : f|_K \in A(K)\}$. We characterize the maximal ideal space and the shilov boundary of these algebras and then discuss the approximation problem among them when X is taken to be fixed but K changes. In particular, we extend the Hartogs-Rosenthal as well as the

Vitushkin's Theorems. Moreover, we show that $P(X, K)$ and $R(X, K)$ are finitely generated under certain conditions.

We also extend the class of Lipschitz algebras on compact metric spaces and study some properties of these Banach function algebras. For example, we determine the maximal ideal space of these extended Lipschitz algebras, and in particular, we show that the analytic Lipschitz algebra $Lip_A(X, K, X)$ is natural when K and X are compact plane sets. Moreover, we extend the class of Dales-Davie algebras of infinitely differentiable functions on compact plane sets and study some properties of these interesting Banach function algebras.

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Chapter 1

Preliminaries

1.1 Notations

Throughout, $\mathbb{N}$ is the set of all positive integers, $\mathbb{R}$ is the set of all real numbers, $\mathbb{C}$ is the set of all complex numbers, and $\mathbb{C}^n$ is the complex n-space.

Let X be a topological space, and $K, T \subseteq X$. We denote the interior of K by $int(K)$, the closure of K by $\overline{K}$, and the boundary of K by $bd(K)$ so that $bd(K) = \overline{K} \cap \overline{(K^c)}$, where K^c is the complement of K. A neighborhood of K is an open subset U of X such that $K \subset U$. We denote the symmetric difference of K and T by $K + T$, and the set of all continuous complex-valued functions on X by $C(X)$.

Let X be a compact Hausdorff space, and let K be a closed subset of X. For $f \in C(X)$, the uniform norm (or sup-norm) of f on K is

$$\|f\|_K = sup\{|f(x)| : x \in K\}.$$

Suppose that f is a continuous function on X, and that $\{f_n\}$ is a sequence of continuous functions on X. Then $\|f_n - f\|_X \to 0$ if and only if $f_n \longrightarrow f$ uniformly on X. A subset S of $C(X)$ is self-adjoint if $\overline{f}$, the complex conjugate of f, belongs to S whenever $f \in S$.

Certain subsets of $\mathbb{C}$ are defined as follows: Let $z \in \mathbb{C}$ and $r > 0$. Then

$$\mathbb{D}(z;r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\} \texttt{ and } \mathbb{T}(z;r) = \{\zeta \in \mathbb{C} : |\zeta - z| = r\}$$

are the open disc and circle, with center z and radius r, respectively. We set $\Delta(z,r) = \mathbb{D}(z;r) \cup \mathbb{T}(z;r) = \overline{\mathbb{D}}(z;r)$.

1.2 Measure Theory

Some basic definitions and results in measure theory are presented in this section. The elementary measure theory, that we assume to be known, can be found in [10] and [25].

Definition 1.2.1. *Let X be a set equipped with a σ-algebra $\mathcal{M}$. A positive measure on $\mathcal{M}$ (or on $(X,\mathcal{M})$) is a function $\mu : \mathcal{M} \longrightarrow [0,+\infty]$ such that:*

(i) $\mu(\emptyset) = 0$,

(ii) If $\{E_j\}_{j=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)$.

Definition 1.2.2. *Let X be a topological space. The σ-algebra generated by the family of open sets in X is called the Borel σ-algebras on X and is denoted by $\mathcal{B}_X$.*

For example, $\mathcal{B}_{\mathbb{R}}$, $\mathcal{B}_{\mathbb{R}^n}$ and $\mathcal{B}_{\mathbb{C}^n}$ are Borel σ-algebra . A measure on $\mathcal{B}_X$ or $(X,\mathcal{B}_X)$ is called a Borel measure.

Definition 1.2.3. *Let $(X,\mathcal{M})$ and $(Y,\mathcal{N})$ be two measurable spaces. The function $f : X \longrightarrow Y$ is measurable if for every $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$.*

If $E \subseteq X$, the characteristic function χ_E is defined by

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

The function $f : X \longrightarrow \mathbb{C}$ is called simple if f is measurable and $range(f)$ is a finite subset of $\mathbb{C}$. If $range(f) = \{z_1, z_2, ..., z_n\}$ and if $E_j = f^{-1}(z_j)$ for $1 \leq j \leq n$ then $f = \sum_{j=1}^{n} z_j \chi_{E_j}$.

If $f : X \longrightarrow \mathbb{C}$ is a measurable function, then there is a sequence $\{\varphi_n\}$ of simple functions such that $0 \leq |\varphi_1| \leq |\varphi_2| \leq ... \leq |\varphi_n| \leq |f|$, $\varphi_n \longrightarrow f$ pointwise, and $\varphi_n \longrightarrow f$ uniformly on any set on which f is bounded [10; Theorem 2.10].

Definition 1.2.4. *Let $(X, \mathcal{M}, \mu)$ be a measure space. We denote the space of all measurable functions from X to $[0, +\infty]$ by L^+. If $\varphi = \sum_{j=1}^{n} a_j \chi_{E_j}$ is a simple function in L^+, we define the integral of φ with respect to μ by $\int \varphi d\mu = \sum_{j=1}^{n} a_j \mu(E_j)$. If $f \in L^+$ we define the integral of f by $\int f d\mu = sup\{\int \varphi d\mu : 0 \leq \varphi \leq f, \varphi\, is\, simple\}$.*

The function $f \in L^+$ is called integrable if the above supremum is finite.

For a real measurable function f we define

$$E_1 = \{x \in X : f(x) > 0\}, E_2 = \{x \in X : f(x) < 0\}.$$

If $f^+ = f\chi_{E_1}$ and $f^- = f\chi_{E_2}$ then $f = f^+ - f^-$ and $f^+, f^- \in L^+$. We define the integral of f by $\int f d\mu = \int f^+ d\mu - \int f^- d\mu$. We say that f is integrable if f^+ and f^- are integrable. It is easy to see that $f^+, f^- \leq |f| \leq f^+ + f^-$.

Next, if f is a complex valued measurable function, we say that f is integrable if $\int |f| d\mu < \infty$, and define

$$\int f d\mu = \int Ref d\mu + i \int Imf d\mu.$$

We denote the space of all integrable functions by $L^1(\mu)$. In general, $L^p(\mu)$ is the set of all measurable functions f such that $|f|^p \in L^1(\mu)$, where $p \geq 1$. Moreover, $\|f\|_p = (\int |f|^p d|\mu|)^{\frac{1}{p}}$ for $f \in L^p(\mu)$.

Theorem 1.2.5. ***(The Dominated Convergence Theorem)****[10; 2.24, page 54]*

Let $\{f_n\}$ be a sequence in $L^1(\mu)$ such that $f_n \longrightarrow f$ a.e. and there exists a non-negative $g \in L^1(\mu)$ such that $|f_n| \leq g$ a.e. for all $n \in \mathbb{N}$. Then $f \in L^1(\mu)$ and $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$.

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. $\mathcal{M} \otimes \mathcal{N}$ is the σ-*algebra* generated by $\mathcal{A} = \{\bigcup_{j=1}^{n} A_j \times B_j : A_j \in \mathcal{M}, B_j \in \mathcal{N}, n \in \mathbb{N}\}$. For $\bigcup_{j=1}^{n} A_j \times B_j \in \mathcal{A}$ we define $(\mu \times \nu)(\bigcup_{j=1}^{n} A_j \times B_j) = \sum_{j=1}^{n} \mu(A_j)\nu(B_j)$. It is easy to see that the product of μ and ν, $\mu \times \nu$, generates a measure on $X \times Y$, for instance, see, [10, VII].

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. If $E \subseteq X \times Y$, for $x \in X$ and $y \in Y$ we define the $x - section$ E_x and the $y - section$ E^y by

$$E_x = \{y \in Y : (x, y) \in E\}, E^y = \{x \in X : (x, y) \in E\}.$$

Also if f is a function on $X \times Y$, we define the $x-section$ f_x and the $y-section$ f^y by

$$f_x(y) = f^y(x) = f(x, y).$$

Theorem 1.2.6. ***(Fubini's Theorem)****[10; 2.37, page 65] Suppose* $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *are* σ*-finite measure spaces. If* $f \in L^1(\mu \times \nu)$*, then* $f_x \in L^1(\nu)$ *for a.e.* $x \in X$*,* $f_y \in L^1(\mu)$ *for a.e.* $y \in Y$*, the functions* $g(x) = \int f_x d\nu$ *and* $h(y) = \int f^y d\mu$ *are in* $L^1(\mu)$ *and* $L^1(\nu)$*, respectively, and*

$$\int f d(\mu \times \nu) = \int \int f(x, y) d\nu(y) d\mu(x) = \int \int f(x, y) d\mu(x) d\nu(y).$$

Definition 1.2.7. *Let* $(X, \mathcal{B}_X, \mu)$ *be a Borel measure space. Let* N *be the union of all open sets* $U \subset X$ *such that* $\mu(U) = 0$*. Then* N *is open, and if* V *is open and* $V \backslash N \neq \emptyset$*, then* $\mu(V) > 0$*. We define support of* μ *as the complement of* N*. In the other words,*

$$supp(\mu) = \cap\{U^c : U \, is \, open \;\; and \;\; \mu(U) = 0\}.$$

The function $\nu : \mathcal{M} \longrightarrow [-\infty, +\infty]$ on the measure space $(X, \mathcal{M})$ is called a signed measure if (i) $\nu(\emptyset) = 0$, (ii) ν assumes at most one of the values $+\infty$ or $-\infty$, and (iii) if $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \nu(E_j)$, where the latter series converges absolutely if $\nu(\bigcup_{j=1}^{\infty} E_j)$ is finite.

If ν is a signed measure then we can write $\nu = \nu^+ - \nu^-$ where ν^+ and ν^- are positive measures.

A complex measure on the measurable space $(X, \mathcal{M})$ is a map $\nu : \mathcal{M} \longrightarrow \mathbb{C}$ such that (i) $\nu(\emptyset) = 0$, and (ii) if $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \nu(E_j)$, where the series on the right is convergent. Note that the convergence is, in fact, absolute convergence.

The Lebesgue measure on $\mathbb{R}$ is the measure m so that, $m([a, b]) = b - a$, the Lebesgue-measure on R^n is the measure m so that, $m([a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n]) = (b_1 - a_1)(b_2 - a_2)...(b_n - a_n)$, the Lebesgue measure on $\mathbb{C}$ is the measure m so that, $m(B(z_0, r)) = \pi r^2$, and the Lebesgue measure on $\mathbb{C}^n$ is the measure m so that, $m(B(z_1, r_1) \times B(z_2, r_2) \times ... \times B(z_n, r_n)) = \pi r_1^2 . \pi r_2^2 ..... \pi r_n^2$.[10]

Definition 1.2.8. *A positive Borel measure ν on $\mathbb{R}^n$ is called regular if (i) $\nu(K) < \infty$ for every compact set K, (ii) $\nu(E) = \inf\{\nu(U) : U \, is\, open, E \subseteq U\}$ and $\nu(E) = \sup\{\nu(K) : K \, is\, compact, K \subseteq E\}$ for every Borel set E. A complex measure is called a Borel regular meagure if $|\mu|$ is a positive Borel measure which is regular.*

For the Lebesgue measure*measure!Lebesgue* m on $\mathbb{R}^n$ we have

$$m(E) = \inf\{m(U) : U \, is\, open, E \subseteq U\}$$
$$= \sup\{m(K) : K \, is\, compact, K \subseteq E\}$$

for every Borel set E in $\mathbb{R}^n$[10].

Definition 1.2.9. *Let $(X, \mathcal{B}_X)$ be a Borel σ-algebra. We denote the set of all regular Borel measures on X by $M(X)$.*

Theorem 1.2.10. *(**Riesz Representation Theorem**)[25] Suppose X is a locally compact Hausdorff space. Then the dual space of $C_0(X)$ is $M(X)$.*

Theorem 1.2.11. *(**Holder's Inequality**)[10; 6.2] Let $1 < p < +\infty$ and $p^{-1} + q^{-1} = 1$ (in other words, $q = \frac{p}{p-1}$). If f and g are measurable functions on X, then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

In particular, if $f \in L^p(\mu)$ and $g \in L^q$, then $fg \in L^1$.

Theorem 1.2.12. *(**Green's Formula**)[22; 5.3.9] Let f be a continuously differentiable function on $\mathbb{R}^2$ which vanishes outside a compact set. Set $f_{\overline{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})$. Then for every α in $\mathbb{C}$,*

$$f(\alpha) = \frac{-1}{\pi} \int \int_{\mathbb{R}^2} \frac{f_{\overline{z}}(x, y)}{x + iy - \alpha} dxdy.$$

Lemma 1.2.13. *[22; 5.3, Lemma 1] Let X be a compact plane set and σ be a positive measure in $M(X)$. Then for almost all α in the plane, $(z - \alpha)^{-1} \in L^1(\sigma)$. Furthermore, if $F(z) = \int_X \frac{d\sigma(\zeta)}{|\zeta - z|}$, then F is integrable (with respect to Lebesgue measure in the plane) over every compact set.*

Lemma 1.2.14. *[22; 5.3, Lemma 2] Let X be a compact set in the plane and $\mu \in M(X)$. If $\int_X \frac{d\mu(\zeta)}{\zeta - z} = 0$ for almost all z, then $\mu = 0$.*

By the above two lemmas we can conclude that if $\mu \in M(X)$ such that $\mu \neq 0$, then there exists a point $z_0 \in X$ such that $\int_X |z - z_0|^{-1} d|\mu|(z) < \infty$ and

$$\int_X (z - z_0)^{-1} d\mu(z) \neq 0.$$

1.3 Complex Analysis

We recall some results from the theory of analytic functions of one complex variable. The elementary complex function theory, that we assume to be known, can be found in [6], [11] and [25].

A subset D of the complex plane is a domain if D is open and connected. For example, open half planes and open disks are domains. An example of an open set which is not a domain is the union of the open upper and lower half planes, *i.e.* $U = \mathbb{C}\backslash\mathbb{R}$.

Definition 1.3.1. *A complex function f is analytic at $a \in \mathbb{C}$ if there exists a neighbourhood U of a such that f is differentiable at each point of U, that is,*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for every $z_0 \in U$.

Theorem 1.3.2. ***(Liouville's Theorem)****[11;10,23] Every bounded entire function is constant.*

Let U be a non-empty open subset of $\mathbb{R}^2$. As in section 1.1, $C(U)$ denotes the space of all continuous functions on U. For $n \in \mathbb{N}$, we define $C^{(n)}(U)$ to be the set of all functions on U whose partial derivatives (with respect to the real variables x, y) up to order n exist and are continuous on U, and we define $C^{(\infty)}(U) = \bigcap_{n=1}^{\infty} C^{(n)}(U)$. The first-order partial differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are defined on $C^{(1)}(U)$ by

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \quad and \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).$$

It is known that a function $f \in C^{(1)}(U)$ is analytic on U if and only if $\frac{\partial f}{\partial \bar{z}} = 0$ on U, and the differential operator $\frac{\partial}{\partial z}$, when applied to an analytic function f, coincides with the usual complex derivative of f, that is, $(\frac{\partial f}{\partial z})(a) = f'(a)$ for all $a \in U$.

If X is a closed subset of $\mathbb{R}^2$ then $C^{(n)}(X)$ is the set of all functions whose partial derivatives up to order n exist in a neighbourhood of X and are continuous on X.

Theorem 1.3.3. ***(Maximum Modulus Principle)****[11; III, page 88] Let $f(z)$ be a complex-valued analytic function on a bounded domain D such that $f(z)$ extends continuously to the boundary $bd(D)$ of D. If $|f(z)| \leq M$ for all $z \in bd(D)$, then $|f(z)| < M$ for all $z \in D$.*

Lemma 1.3.4. [*11; XIII, page 343*] *Let K be a compact subset of the complex plane, let U be a connected open subset of the extended complex plane $\mathbb{C}^*$ disjoint from K, and $z_0 \in U$. Every rational function with poles in U can be approximated uniformly on K by rational functions with poles at z_0.*

By approximating with rational functions and then using the above lemma to translate the poles, we obtain immediately the following sharper version of Runge's theorem.

Theorem 1.3.5. [*11; XIII, page 344*] *Let K be a compact subset of the complex plane, and suppose that $f(z)$ is analytic on an open set containing K. Let S be a subset of $\mathbb{C}^*\backslash K$ such that each connected component of $\mathbb{C}^*\backslash K$ contains a point of S. Then $f(z)$ can be approximated uniformly on K by rational functions with poles in S.*

1.4 Functional Analysis

In this section, we shall gather together some results of functional analysis for easy reference.

Throughout this thesis, all vector spaces are assumed to be over the complex field $\mathbb{C}$. Terms and concepts of basic real and functional analysis, which we have not defined or discussed, can be found in [5], [7] and [24].

A topological vector space is a vector space X together with a topology such that with respect to this topology

(i) every point of X is a closed set,

(ii) the map $X \times X \longrightarrow X$ defined by $(x, y) \longmapsto x + y$ is continuous,

(iii) the map $\mathbb{C} \times X \longrightarrow X$ defined by $(\alpha, x) \longmapsto \alpha x$ is continuous.

A topological vector space X is called locally convex, if there is a local base at $0 \in X$ whose members are convex.

A vector space X is said to be a normed space if for every $x \in X$ there is associated a non-negative real number $\|x\|$, called the norm of x, such that for every $x, y \in X$ and any scalar α,

(i) $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$,

(iii) $\|\alpha x\| = |\alpha|\|x\|$.

If $\|.\|$ satisfies only (ii) and (iii), then it is called a *semi−norm* on X. Every normed space $(X, \|.\|)$ is a locally convex topological vector space. A complete normed space is called a Banach space.

For a topological vector space X, a linear functional on X is a linear mapping of X into $\mathbb{C}$. It is known that a linear functional f on X is continuous if and only if f is continuous at 0 and if and only if f is bounded in some neighborhood V of 0 [24; Theorem 1.18]. For a topological vector space X, the set of all continuous linear functionals on X is denoted by X^* and is called the dual space of X. Clearly, X^* is a vector space, and if $(X, \|.\|)$ is a normed space then X^*, equipped with the norm

$$\|f\| = \sup\{|f(x)| : x \in X, \|x\| \leq 1\},$$

is a Banach space [24; Theorem 4.1].

Let X be a topological vector space with the dual space X^*. Then every $x \in X$ induces a linear functional $\widehat{x}$ on X^* defined by $\widehat{x}(f) = f(x)$ and $\widehat{X} = \{\widehat{x} : x \in X\}$ separates the points of X^*. The w^*-topology(or weak topology induced by $\widehat{X}$) on X^* is the weakest topology on X^* under which every $\widehat{x}$ is a continuous linear functional on X^*. It is known that X^* with the w^*-topology is a locally convex topological vector space whose dual space is $\widehat{X}$ [24; Theorem 3.10].

1.5 Banach Algebras

We now turn to the definitions and fundamental properties of Banach Algebras. The elementary theory of Banach Algebras can be found in many texts; for example, [5], [16], [24] and [32].

An algebra is a complex vector space A together with a multiplication, called an algebra product, $A \times A \longrightarrow A$, $(a, b) \longmapsto ab$, which is associative and respects the vector operations:

$$a(b+c) = ab + ac, \quad a(bc) = (ab)c,$$

$$(b+c)a = ba + ca, \quad \lambda(ab) = (\lambda a)b = a(\lambda b),$$

for all $a, b, c \in A$ and $\lambda \in \mathbb{C}$. We say that A is commutative if

$$ab = ba \quad (a, b \in A).$$

A subalgebra of A is a linear subspace B of A such that B is closed under multiplication, that is, $ab \in B$ whenever $a, b \in B$. We say that an algebra A is unital if A has an identity, i.e., there is an element $1 \in A$ such that $1a = a1 = a$, for all $a \in A$. An element $a \in A$ is invertible if it has an inverse, that is, if there exists an element $b \in A$ such that $ab = ba = 1$. If $a \in A$ is invertible, then the inverse of a is unique and is denoted by a^{-1}. We denote the set of all invertible elements of an algebra A by $Inv(A)$.

Let A be an algebra. An algebra norm on A is a norm $\|.\|$ on A such that

$$\|ab\| \leq \|a\|\|b\| \quad (a, b \in A).$$

If $\|.\|$ is an algebra norm on A, then $(A, \|.\|)$ is called a normed algebra. A complete normed algebra is a Banach algebra.

For a normed algebra A, the multiplication $(a, b) \longmapsto ab$, $A \times A \longrightarrow A$ is continuous, i.e., if $a_n \longrightarrow a$ and $b_n \longrightarrow b$ then $a_n.b_n \longrightarrow ab$. Conversely, according to Theorem 10.2 in [24], if $(A, \|.\|)$ is a Banach space as well as an algebra with identity 1, in which multiplication is continuous, then there is an algebra norm $\|.\|'$ on A which is equivalent to $\|.\|$ and which makes A into a Banach algebra such that $\|1\|' = 1$.

Let $\|.\|$ be a norm on an algebra A, and suppose that there exists $C > 0$ such that

$$\|ab\| \leq C\|a\|\|b\| \quad (a, b \in A).$$

Set $\|a\|' = C\|a\|(a \in A)$. Then $\|.\|'$ is an algebra norm on A which is equivalent to $\|.\|$, and $(A, \|.\|')$ is a Banach algebra whenever $(A, \|.\|)$ is complete.

A normed algebra $(A, \|.\|)$ is called a unital normed algebra if A has an identity 1 and $\|1\| = 1$. If $(A, \|.\|)$ is a Banach algebra with identity, then there is a norm $\|.\|'$, equivalent to $\|.\|$, such that $(A, \|.\|')$ is a unital Banach algebra [8; Proposition 2.1.9]. Thus we may always suppose that a Banach algebra with an identity is a unital Banach algebra.

Let A be a unital Banach algebra, and let $a \in A$. The spectrum $Sp(a, A)$ of a is the set of all complex numbers λ such that $\lambda 1 - a$ is not invertible in A. In other words,

$$Sp(a, A) = \{\lambda \in \mathbb{C} : \lambda 1 - a \notin Inv(A).\}$$

The spectral radius of a is the number

$$\rho(a) = \sup\{|\lambda| : \lambda \in Sp(a, A)\}.$$

If there is no ambiguity, we write $Sp(a)$ instead of $Sp(a, A)$.

Theorem 1.5.1. *[24; Theorem 10.13] If A is a unital Banach algebra and $a \in A$, then*

(i) the spectrum $Sp(a)$ of a is non-empty and compact.

(ii) the spectral radius $\rho(a)$ of a satisfies

$$\rho(a) = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}} = \inf\{\|a^n\|^{\frac{1}{n}} : n \in \mathbb{N}\}.$$

Whether an element of A is or is not invertible in A is a purely algebraic property. The spectrum and the spectral radius of an element $a \in A$ are thus defined in terms of the algebraic structure of A, regardless of any topological considerations. On the other hand, $\lim_{n \longrightarrow \infty} \|a^n\|^{\frac{1}{n}}$ depends obviously on metric properties of A. This is one of the remarkable features of the spectral radius formula.

In the following let X be a compact Hausdorff space. For $A \subset C(X)$, the annihilator of A is

$$A^{\perp} = \Big\{\mu \in M(X) : \int f d\mu = 0, \; for \; every \; f \in A\Big\}.$$

Proposition 1.5.2. *Let A and B be closed subalgebras of $C(X)$. Then $A \subseteq B$ if and only if $B^{\perp} \subseteq A^{\perp}$.*

Proof. By the Hahn-Banach theorem it is immediate. □

Theorem 1.5.3. ***(Stone-Weierstrass)****[22; 3.2.3] Suppose A is a closed subalgebra of $C(X)$ which separates the points of X and is self-adjoint. Then either $A = C(X)$ or there is a point $x \in X$ such that A is the maximal ideal $I_x = \{f \in C(X) : f(x) = 0\}$.*

1.6 Commutative Banach Algebras and Gelfand Transforms

In this section, we give some results about commutative Banach algebras. An ideal in a commutative algebra A is a subspace $I \subseteq A$ such that $ab \in I$ whenever $a \in A$ and $b \in I$. If $I \neq A$ then I is a proper ideal in A. A proper ideal M in A is said to be maximal if for every ideal I in A the condition $M \subseteq I \subseteq A$ implies that $I = M$ or $I = A$. No proper ideal of A contains any invertible element of A. If A is a commutative unital algebra, an argument using the Zorn's lemma shows that every proper ideal of A is contained in a maximal ideal of A.

Let A be a commutative Banach algebra. Since the algebra product $(a, b) \longmapsto ab$ is continuous, if I is an ideal [a proper ideal] in A, then its closure $\overline{I}$ is also an ideal [a proper ideal] in A. Moreover, if M is a maximal ideal in A, then M is closed.

Theorem 1.6.1. [*2; Theorem 3.1.3*] *Let A be a commutative algebra with identity 1. Then the following sets are identical:*

(i) the intersection of all maximal ideals in A,

(ii) the set of all elements $a \in A$ such that $1 - ab$ is invertible in A, for all $b \in A$.

The subset of A having the properties of the above theorem is called the radical of A, denoted by $radA$. Since any intersection of closed ideals in a Banach algebra A is also a closed ideal, $radA$ is a closed ideal in A. The algebra A is said to be semi-simple if $radA = 0$.

Let A and B be algebras. A linear map $\theta : A \longrightarrow B$ is an algebra homomorphism if $\theta(ab) = \theta(a)\theta(b)$, for all $a, b \in A$. If θ is a bijection, then θ is an algebra isomorphism. The kernel of θ, denoted by $ker\theta$, is the set $\{a \in A : \theta(a) = 0\}$. Clearly, $ker\theta$ is an ideal in A. If A and B are normed algebras, then $ker\theta$ is a closed ideal in A if θ is continuous.

Let A be a Banach algebra. A complex homomorphism (or character) on A is a nonzero algebra homomorphism $h : A \longrightarrow \mathbb{C}$.

It is well-known that, for every Banach algebra A, every complex homomorphism h on A is continuous and $\|h\| \leq 1$. If A is a unital Banach algebra, then $h(1) = \|h\| = 1$, moreover, $h(a) \neq 0$ for every invertible element a. We denote the set of all complex homomorphisms on A by M_A.

Theorem 1.6.2. *[24; Theorem 11.5] Let A be a commutative unital Banach algebra. Then*

(i) if $h \in M_A$, the kernel of h is a maximal ideal of A,

(ii) every maximal ideal of A is the kernel of some $h \in M_A$,

(iii) an element $a \in A$ is invertible if and only if $h(a) \neq 0$ for all $h \in M_A$,

(iv) $Sp(a) = \{h(a) : h \in M_A\}$, for all $a \in A$,

(v) $\rho(a) = \sup\{|h(a)| : h \in M_A\}$, for all $a \in A$.

Let A be a commutative unital Banach algebra. For every $a \in A$ the mapping $\widehat{a} : h \longmapsto h(a)$, $M_A \longrightarrow \mathbb{C}$ is the Gelfand transform of a, and the mapping $a \longmapsto \widehat{a}$, $A \longrightarrow \widehat{A}$, is the Gelfand transform of A, where $\widehat{A} = \{\widehat{a} : a \in A\}$. The Gelfand topology of M_A is the weak topology induced by $\widehat{A}$, that is, the weakest topology that makes every $\widehat{a}$ continuous. A basic neighbourhood of $\phi \in M_A$ is of the form

$$V(\phi; x_1, ..., x_n; \varepsilon) = \{\psi \in M_A : |\psi(x_i) - \phi(x_i)| < \varepsilon \ for \ i = 1, ..., n\},$$

for arbitrary positive integer n, elements $x_1, ..., x_n \in A$, and $\varepsilon > 0$. Thus a net $\{\phi_\alpha\}$ in M_A converges to ϕ in the Gelfand topology of M_A if and only if $\phi_\alpha(x) \longrightarrow \phi(x)$ for all $x \in A$.

In fact, the Gelfand topology on M_A is the relative topology on M_A induced by the w^*-topology on A^*. If A is a unital Banach algebra, then M_A is a nonempty

w^*-closed subset of $\{f \in A^* : \|f\| \leq 1\}$, the unit ball of A^*. Therefore, by Banach-Alaoglu Theorem, M_A is a w^*-compact space.

Let A be a unital Banach algebra. Since there is a one-one correspondence between the maximal ideals of A and the complex homomorphism on A, M_A equipped with its Gelfand topology, is called the maximal ideal space(or character space) of A.

Theorem 1.6.3. [*5; Theorem 4.17*] *Let A be a unital commutative Banach algebra. Then*

(i) M_A is a compact Hausdorff space,

(ii) the mapping $a \longmapsto \widehat{a}$ is an algebra homomorphism of A into $C(M_A)$,

(iii) $Sp(a) = \widehat{a}(M_A) = \{h(a) : h \in M_A\}$, for all $a \in A$,

(iv) $\rho(a) = \|\widehat{a}\| = \sup\{|h(a)| : h \in M_A\}$, for all $a \in A$.

We regard $\widehat{A}$ as a subalgebra of $C(M_A)$. For every $a \in A$

$$\begin{aligned}\|\widehat{a}\| &= \sup\{|h(a)| : h \in M_A\} = \sup\{|\lambda| : \lambda \in Sp(a)\} \\ &= \inf\{\|a^n\|^{\frac{1}{n}} : n = 1, 2, 3, ...\} \leq \|a\|\end{aligned}$$

1.7 Banach Function Algebras

In this section, we recall some elementary properties of Banach function algebras.

Let X be a compact Hausdorff space. Then $C(X)$, the space of all continuous complex-valued functions on X with the pointwise multiplication, is a commutative unital algebras, and the uniform norm $\|.\|_X$ is a complete algebra norm on $C(X)$ so that $(C(X), \|.\|_X)$ is a commutative unital Banach algebra. It is known that the maximal ideal space of $C(X)$ is homeomorphic to X. See, for example, [24;section 11.13 (a)] or [12; I.3.1].

Definition 1.7.1. *Let X be a compact Hausdorff space. A function algebra on X is a subalgebra A of $C(X)$ which contains the constant functions and separates the points of X. If there is a norm $\|.\|$ on A such that $(A, \|.\|)$ is a normed algebra,*