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Gorenstein Dimensions in Commutative Algebra

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Abstract

The central homological notions of commutative algebra are the classical homological dimensions. These have been proved to be important invariants for modules and complexes of modules over commutative (noetherian) rings. This thesis deals with the refinements of these invariants, the Gorenstein dimensions. It contains some new results on Gorenstein dimensions, as generalizations of some of the well-known results on classical homological dimensions. Our results have two features, they present relations between the Gorenstein dimensions and the grade of modules and complexes, and, they treat the Gorenstein dimensions under base change.

Preface

The first serious attempt to use *homological methods* in commutative algebra was by Arthur Cayley in *elimination theory*, in 1848. But the homological methods have found their way in the subject and have become a very important tool in commutative algebra since the mid 1950's.

The *Gorenstein dimensions* which are studied in this manuscript, are refinements of the central notions in the classical homological algebra, namely, the *projective dimension*, the *injective dimension* and the *flat dimension.*

This thesis is organized in five sections. In the first section, the background and the basic definitions and results of the theory are presented. The reader who is familiar with the theory of Gorenstein dimensions, can easily skip this section. The only exception is an outline of the main results of the whole thesis in the first section. These are distinguished by underlining.

Other sections contain some new results on Gorenstein dimensions. In sections 2, 3 and 4, some relations between the Gorenstein dimensions and the *grade* of modules and complexes of modules are studied. Section 2 is based on [36] and studies the Cohen-Macaulayness of tensor products. In section 3 we have generalized the results of [46] to complexes of modules. The results of this section can be considered as the generalizations of some well-known results on perfect modules. In section 3, a generalized definition of the grade of a module is given. The basis of this section is [47]. Furthermore some new results on Gorenstein flat dimensions are presented.

The core of this manuscript is the fifth section, where some generalizations of the so-called change of ring results to Gorenstein dimensions are studied. The main results of this section deal with the relations between Gorenstein dimensions of complexes of modules over different base rings. Some of the results of this section have appeared in [37].

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Contents

1 Introduction

In this section we fix the notations for the other parts of the manuscript. The main purpose of this section is to give a review of the background and the bases of the theory of the Gorenstein dimensions. For details and proofs of the results, we have usually given references to recent text books in the subject. Furthermore we have pointed out the papers where the theorems have originally appeared.

1.1 Notation and Terminology

In this thesis, all rings are assumed to be non-trivial and commutative. A ring R is said to be *local* if it is noetherian with a unique maximal ideal. We say that the R-module M is *finite* if it is a finitely generated R-module. Any unreferenced material in this section can be found in [25] or [12].

Definition 1.1.1 *Let R be a ring. An R-*complex *X is a sequence of R-modules X_ℓ and R-homomorphisms ∂_ℓ^X, $\ell \in \mathrm{Z}$,*

$$X = \dots \longrightarrow X_{\ell+1} \xrightarrow{\partial_{\ell+1}^X} X_\ell \xrightarrow{\partial_\ell^X} X_{\ell-1} \longrightarrow \dots$$

such that $\partial_\ell^X \partial_{\ell+1}^X = 0$ for all $\ell \in \mathrm{Z}$. X_ℓ and ∂_ℓ^X are called the module in degree ℓ *and the ℓ-th* differential *of X, respectively. An R-module M is considered as the complex $0 \longrightarrow M \longrightarrow 0$ with M in degree 0.*

For an R-complex X and an integer ℓ, the following notations are used.

$$\mathrm{Z}_\ell^X = \mathrm{Ker}\partial_\ell^X$$
$$\mathrm{B}_\ell^X = \mathrm{Im}\partial_{\ell+1}^X$$
$$\mathrm{C}_\ell^X = \mathrm{Coker}\partial_{\ell+1}^X$$

It is clear that $\mathrm{B}_\ell^X \subseteq \mathrm{Z}_\ell^X$. *The residue class module*

$$\mathrm{H}_\ell(X) = \mathrm{Z}_\ell^X / \mathrm{B}_\ell^X$$

is called the homology module in degree ℓ. *The* homology complex *is the R-complex* $\mathrm{H}(X)$ *which has* $\mathrm{H}_\ell(X)$ *as the module in degree* ℓ *and zero differentials in all degrees.*

Definition 1.1.2 *A* morphism $\alpha : X \longrightarrow Y$ *of R-complexes is a family* $\alpha = (\alpha_\ell)_{\ell \in \mathrm{Z}}$ *of R-homomorphism* $\alpha_\ell : X_\ell \longrightarrow Y_\ell$ *satisfying* $\partial_\ell^Y \alpha_\ell - \alpha_{\ell-1}\partial_\ell^X = 0$, *for all* $\ell \in \mathrm{Z}$. *A morphism* $\alpha : X \longrightarrow Y$ *is said to be a* quasi-isomorphism *if it induces an isomorphism in homology. Quasi-isomorphisms are indicated by the symbol* $\simeq$ *above their arrows.*

Two R-complexes X and Y are said to be equivalent, $X \simeq Y$, *if and only if there exists an R-complex Z and two quasi-isomorphisms* $X \xrightarrow{\simeq} Z \xleftarrow{\simeq} Y$.

Definition 1.1.3 *The* supremum, *the* infimum, *and the* amplitude *of an R-complex X are defined as*

$$\begin{aligned} \sup X &= \sup\{\ell \in \mathrm{Z} \mid \mathrm{H}_\ell(X) \neq 0\}, \\ \inf X &= \inf\{\ell \in \mathrm{Z} \mid \mathrm{H}_\ell(X) \neq 0\}, \\ \mathrm{amp}_R X &= \sup X - \inf X. \end{aligned}$$

X is said to be homologically trivial *if* $\mathrm{H}(X) = 0$; *for such complexes* $\inf X = +\infty$ *and* $\sup X = -\infty$, *by convention.*

Definition 1.1.4 *The notation* $\mathcal{C}(R)$ *denotes the category of all R-complexes and all morphisms of R-complexes.*

The full subcategories $\mathcal{C}_\sqsubset(R), \mathcal{C}_\sqsupset(R)$, $\mathcal{C}_\square(R)$, *and* $\mathcal{C}_0(R)$ *consist of all R-complexes X with* $\mathrm{H}_\ell(X) = 0$ *for* $\ell \gg 0$, $\ell \ll 0$, $|\ell| \gg 0$ *and* $\ell \neq 0$, *respectively. The full subcategories* $\mathcal{C}_{(\sqsubset)}(R), \mathcal{C}_{(\sqsupset)}(R)$, $\mathcal{C}_{(\square)}(R)$ *consist of all R-complexes X such that* $\mathrm{H}(X)$ *belongs to* $\mathcal{C}_\sqsubset(R), \mathcal{C}_\sqsupset(R)$, $\mathcal{C}_\square(R)$, *respectively.*

We also consider the following full subcategories of $\mathcal{C}(R)$.

$\mathcal{C}^{(f)}(R)$: *Complexes of finite homology modules*;

$\mathcal{C}^{P}(R)$: *Complexes of projective modules*;

$\mathcal{C}^{I}(R)$: *Complexes of injective modules*;

$\mathcal{C}^{F}(R)$: *Complexes of flat modules*;

$\mathcal{C}^{L}(R)$: *Complexes of finite free modules*.

Superscripts and subscripts are freely mixed to produce new notations.

Definition 1.1.5 *The notations* $\mathbf{R}\mathrm{Hom}_R(-,-)$ *and* $- \otimes_R^{\mathbf{L}} -$ *are used for, respectively, the right derived functor of the homomorphism functor and the left derived functor of the tensor product functor of R-complexes.*

Definition 1.1.6 *The subcategories* $\mathcal{P}(R), \mathcal{I}(R)$, *and* $\mathcal{F}(R)$ *are defined as follows.*

$$X \in \mathcal{P}(R) \Leftrightarrow \exists P \in \mathcal{C}^{P}_{\square}(R) : X \simeq P;$$
$$X \in \mathcal{I}(R) \Leftrightarrow \exists I \in \mathcal{C}^{I}_{\square}(R) : X \simeq I;$$
$$X \in \mathcal{F}(R) \Leftrightarrow \exists F \in \mathcal{C}^{F}_{\square}(R) : X \simeq F.$$

The projective dimension, injective dimension *and* flat dimension *of* $X \in \mathcal{C}_{(\square)}(R)$ *is defined as*

$$\mathrm{pd}_R X = \inf\{\sup\{\ell \in \mathrm{Z} \mid P_\ell \neq 0\} \mid X \simeq P \in \mathcal{C}^{P}_{\sqsupset}(R)\},$$
$$\mathrm{id}_R X = \inf\{\sup\{\ell \in \mathrm{Z} \mid I_{-\ell} \neq 0\} \mid X \simeq I \in \mathcal{C}^{I}_{\sqsubset}(R)\} \text{ and}$$
$$\mathrm{fd}_R X = \inf\{\sup \mid \ell \in \mathrm{Z} \mid F_\ell \neq 0\} \mid X \simeq F \in \mathcal{C}^{F}_{\sqsupset}(R)\}.$$

Theorem 1.1.7 *For* $X \in \mathcal{P}(R)$, $Y \in \mathcal{I}(R)$, *and* $Z \in \mathcal{F}(R)$, *the following equalities hold.*

$$\begin{aligned}
\mathrm{pd}_R X &= \sup\{\inf U - \inf(\mathbf{R}\mathrm{Hom}_R(X,U)) \mid U \in \mathcal{C}_{(\square)}(R) \wedge U \not\simeq 0\} \\
&= \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(X,T)) \mid T \in \mathcal{C}_0(R)\} \\
\mathrm{id}_R Y &= \sup\{-\sup U - \inf(\mathbf{R}\mathrm{Hom}_R(U,Y)) \mid U \in \mathcal{C}_{(\square)}(R) \wedge U \not\simeq 0\} \\
&= \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(T,Y)) \mid T \in \mathcal{C}_0(R) \text{ cyclic}\} \\
\mathrm{fd}_R Z &= \sup\{\sup(U \otimes_R^{\mathbf{L}} Z) - \sup U \mid U \in \mathcal{C}_{(\square)}(R) \wedge U \not\simeq 0\} \\
&= \sup\{\sup(T \otimes_R^{\mathbf{L}} Z) \mid T \in \mathcal{C}_0(R) \text{ cyclic}\}
\end{aligned}$$

Theorem 1.1.8 *Let R and S be Q-algebras. Then there are identities of equivalence classes of Q-complexes, as follows.*

Commutativity. *If $X \in \mathcal{C}_{(\sqsupset)}(R)$ and $Y \in \mathcal{C}(R)$, then*

$$X \otimes_R^{\mathbf{L}} Y = Y \otimes_R^{\mathbf{L}} X.$$

Associativity. *If $X \in \mathcal{C}_{(\sqsupset)}(R), Y \in \mathcal{C}(R,S)$ and $Z \in \mathcal{C}_{(\sqsupset)}(S)$, then*

$$(X \otimes_R^{\mathbf{L}} Y) \otimes_S^{\mathbf{L}} Z = X \otimes_R^{\mathbf{L}} (Y \otimes_S^{\mathbf{L}} Z).$$

Adjointness. *If $X \in \mathcal{C}_{(\sqsupset)}(R), Y \in \mathcal{C}(R,S)$ and $Z \in \mathcal{C}_{(\sqsubset)}(S)$, then*

$$\mathbf{R}\mathrm{Hom}_S(X \otimes_R^{\mathbf{L}} Y, Z) = \mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\mathrm{Hom}_S(Y,Z)).$$

Tensor Evaluation. *If R is a notherian ring and $X \in \mathcal{C}_{(\sqsupset)}^{(f)}(R)$, $Y \in \mathcal{C}_{(\square)}(R,S)$ and $Z \in \mathcal{C}_{(\sqsupset)}(S)$, then*

$$\mathbf{R}\mathrm{Hom}_R(X,Y) \otimes_S^{\mathbf{L}} Z = \mathbf{R}\mathrm{Hom}_R(X, Y \otimes_S^{\mathbf{L}} Z),$$

provided that $X \in \mathcal{P}^{(f)}(R)$ or $Z \in \mathcal{F}(S)$.

Hom Evaluation. *If R is a noetherian ring and $X \in \mathcal{C}_{(\sqsupset)}^{(f)}(R)$, $Y \in \mathcal{C}_{(\square)}(R,S)$, and $Z \in \mathcal{C}_{(\sqsubset)}(S)$, then*

$$X \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_S(Y,Z) = \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(X,Y),Z),$$

provided that $X \in \mathcal{P}^{(f)}(R)$ or $Z \in \mathcal{I}(S)$.

Definition 1.1.9 *Let $\mathfrak{a}$ be an ideal of R and $\mathbf{a} = a_1, \ldots, a_t$ be a finite set of generators for $\mathfrak{a}$, then* $\mathfrak{a}$-depth *of $X \in \mathcal{C}(R)$ is defined to be the number*

$$\operatorname{depth}_R(\mathfrak{a}, X) = t - \sup(\mathrm{K}(\mathbf{a}) \otimes_R X).$$

(cf. [33])

Recall that the Koszul complex on an element $x \in R$ is the complex

$$\mathrm{K}(x) = 0 \longrightarrow R \xrightarrow{.x} R \longrightarrow 0$$

concentrated in degrees 1 and 0. For a sequence $\mathbf{x} = x_1, \dots, x_n$ of elements of R the Koszul complex $\mathrm{K}(\mathbf{x}) = \mathrm{K}(x_1, \dots, x_n)$ is the tensor product $\mathrm{K}(x_1) \otimes_R \dots \otimes_R \mathrm{K}(x_n)$.

Definition 1.1.10 *For $X \in \mathcal{C}_{(\sqsupset)}(R)$ and $Y \in \mathcal{C}_{(\sqsubset)}(R)$, the* grade *of X and Y is defined as follows.*

$$\mathrm{grade}_R(X, Y) = -\sup(\mathbf{R}\mathrm{Hom}_R(X, Y))$$

(cf. [46])

Definition 1.1.11 *If $(R, \mathfrak{m}, k)$ is a local ring and $X \in \mathcal{C}_{(\sqsubset)}(R)$, then the* depth *of X is defined as*

$$\mathrm{depth}_R X = \mathrm{depth}_R(\mathfrak{m}, X) = \mathrm{grade}_R(k, X).$$

If $X \in \mathcal{C}^{(f)}_{(\sqsupset)}(R)$ and $Y \in \mathcal{C}_{(\sqsubset)}(R)$, then

$$\mathrm{grade}_R(X, Y) = \inf\{\mathrm{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} + \inf X_\mathfrak{p} | \, \mathfrak{p} \in \mathrm{Spec} R\}.$$

It is proved ([14]) that if $X \in \mathcal{C}_{(\sqsubset)}(R)$, then

$$\mathrm{depth}_R(\mathfrak{a}, X) = \mathrm{grade}_R(R/\mathfrak{a}, X).$$

Definition 1.1.12 *Let R be a noetherian ring. The* (Krull) dimension *of $X \in \mathcal{C}_{(\sqsupset)}(R)$ is defined as*

$$\dim_R X = \sup\{\dim(R/\mathfrak{p}) - \inf X_\mathfrak{p} | \, \mathfrak{p} \in \mathrm{Spec} R\}.$$

Definition 1.1.13 *If R is a local ring, then the* Cohen-Macaulay defect *of a homologically bounded complex X with finite depth is defined as*

$$\mathrm{cmd}_R X = \dim_R X - \mathrm{depth}_R X.$$

The complex $X \in \mathcal{C}^{(f)}_{(\square)}(R)$ is said to be Cohen-Macaulay *if $\mathrm{cmd}_R X = 0$.*

Definition 1.1.14 *Let R be a local ring. An R-complex D is a* dualizing complex *for R if D belongs to $\mathcal{I}^{(f)}(R)$ and the natural morphism $R \longrightarrow \mathbf{R}\mathrm{Hom}_R(D, D)$ is invertible.*

It is well-known that the equality $\operatorname{cmd} R = \operatorname{amp} D$ always holds.

1.2 Auslander's G-dimension

The *projective dimension* has proved to be a very important invariant for modules over commutative noetherian rings. Recalling the next two classical results, both proved in 1950's, will suffice to illustrate this importance. The first theorem is due to Auslander and Buchsbaum [4] and Serre [42] (cf. [10], 2.2.7).

Theorem 1.2.1 *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring. The following are equivalent.*

(i) R is regular (R has finite global dimension).

(ii) $\mathrm{pd}_R k < \infty$.

(iii) $\mathrm{pd}_R M < \infty$ *for all finite R-modules M.*

(iv) $\mathrm{pd}_R M < \infty$ *for all R-modules M.*

The second result was also proved by Auslander and Buchsbaum [6] and is well-known as *Auslander-Buchsbaum formula* (cf. [10], 1.3.3).

Theorem 1.2.2 *Let M be a finite module over a commutative noetherian local ring R. If M has finite projective dimension, then*

$$\mathrm{pd}_R M = \operatorname{depth} R - \operatorname{depth}_R M.$$

The *Gorenstein dimension* of a finite module M over a commutative noetherian ring R, $\text{G-dim}_R M$, was introduced in 1967 by Auslander [2]. This relative homological dimension is a finer invariant than projective dimension in the sense that there is always an inequality $\text{G-dim}_R M \leq \mathrm{pd}_R M$ and equality holds when $\mathrm{pd}_R M$ is finite. Furthermore, Gorenstein dimension shares many of the nice properties of projective dimension. Some of these properties will be stated in this thesis. But first let us review the definitions and basic properties.

Convention. In the rest of this section, R is a commutative noetherian ring.

Definition 1.2.3 *A non-zero finite R-module M belongs to the G-*class *of R, $G(R)$, if and only if*

(i) $\mathrm{Ext}^i_R(M, R) = 0$ *for* $i > 0$,

(ii) $\mathrm{Ext}^i_R(\mathrm{Hom}_R(M, R), R) = 0$ *for* $i > 0$, *and*

(iii) The canonical map $M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ *is an isomorphism.*

Modules in $G(R)$ are also called modules of *G-dimension zero.*

Now a G-resolution for a finite R-module can be constructed using modules in the G-class.

Definition 1.2.4 *A G-resolution of a finite R-module M is a sequence of modules in $G(R)$,*

$$\ldots \longrightarrow G_\ell \longrightarrow G_{\ell-1} \longrightarrow \ldots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow 0$$

which is exact in G_ℓ for $\ell > 0$ and has $G_0/Im(G_1 \longrightarrow G_0) \cong M$.

The resolution is said to be of length n if $G_n \neq 0$ and $G_\ell = 0$ for $\ell > n$.

Remark 1.2.5 *It is clear that every finite free R-module is in $G(R)$ and it is easy to prove that every finite projective R-module is in $G(R)$, too. Thus every finite module has a G-resolution.*

Definition 1.2.6 *Let M be a non-zero finite R-module. For $n \in \mathbb{N}_0$ we say that M has G-dimension at most n, and we write* $\text{G-dim}_R M \leq n$, *if and only if M has a G-resolution of length n. If M does not have any G-resolution of finite length, then we say that it* *has infinite G-dimension and write* $\text{G-dim}_R M = \infty$. *The G-dimension of zero module is defined to be* $-\infty$.

Auslander has proved [2] that G-dimension of a finite module can be computed in terms of the vanishing of the Ext functors.

Theorem 1.2.7 *([12], 1.2.7) Let M be a finite R-module of finite G-dimension. Then*

$$\text{G-dim}_R M = \sup\{i \mid \text{Ext}^i_R(M, R) \neq 0\}.$$

At the beginning of this section, we pointed out to some important theorems about projective dimension (2.1 and 2.2). The next two theorems, due to Auslander and Bridger [3], show that the Gorenstein dimension has also similar properties.

Theorem 1.2.8 *([12], 1.4.9) Let $(R, \mathfrak{m}, k)$ be a local ring. Then the following are equivalent.*

(i) R is Gorenstein (R has finite self-injective dimension).

(ii) $\text{G-dim}_R k < \infty$.

(iii) $\text{G-dim}_R M < \infty$ *for all finite R-modules M.*

Note that this theorem is analogous to the theorem 1.2.1, but since the Gorenstein dimension is only defined for finite modules, the analogy is not complete. This will be made up in the next section.

Theorem 1.2.9 *([12], 1.4.8) Let M be a finite module over a local ring R. If M has finite G-dimension, then*

$$\text{G-dim}_R M = \text{depth} R - \text{depth}_R M.$$