IN THE NAME OF ALLAH

CS RINGS AND NAKAYAMA PERMUTATIONS

BY

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To:

My Belove Father and Mother

and

My Dear Wife

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ABSTRACT

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In chapter I we state basic definitions, theorems and lemmas that are necessary in the other chapters.

In chapter II we define Kasch rings and continuous rings and prove some results of mininjective rings. And we show that if R is a semiperfect left continuous ring with essential left socle, then R admits a Nakayama permutation of its basic set of primitive idempotents.

In chapet III we show that a ring R is left CS and the dual of every simple right R-module is simple if and only if R is semiperfect left continuous with $Soc_R R = Soc R_R \overset{ess}{\subseteq} {}_R R$. Moreover in this case R is also left Kasch and admits a Nakayama permutation of its basic set of primitive idempotents. We also characterize left PF rings and we show that a ring R is quasi-Frobenius if and only if R is right Kasch and ${}_R R^{(w)}$ is CS.

In chapter IV we provide some examples that satisfy or do not satisfy in some concepts of previous chapters.

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CHAPTER I

BASIC DEFINITIONS AND RESULTS

Throughout this dissertation R will denote an associative ring with unity and all modules are unitary R-modules. We write $A \subseteq B (A \subset B)$ to mean A is a (proper) submodule of B. The notation $A \overset{ess}{\subseteq} B$ and $C \subseteq^{\oplus} D$ will mean A is an essential submodule of B and C is a direct summand of D. If M_R is a right R-module, we will denote by $J(M), Z(M), Soc(M)$ and $E(M)$ the Jacobson radical, the singular submodule, the socle and the injective hull of M, respectively. The left (respectively right) annihilator of a subset X of R is denoted by $\ell(X)$ (respectively, $r(X)$) and is denoted by

$$ann_\ell(x) = \ell(x) = \{r \in R |\, rx = 0,\ \forall x \in X\}.$$

$(ann_r(x) = r(x) = \{r \in R |\, xr = 0,\ \forall x \in X\})$.

We will write $M^{(k)}$ and M^k to indicate the direct sum of k-copies of M and the direct product of k-copies of M, respectively. We will indicate by $M_R^* = Hom_R(M_R, R_R)$ the dual of right R-module M.

1.1. Essential and Superfluous Submodules

Definition 1.1.1. Let M be a module. A submodule K of M is essential (or large) in M, abbreviated $K \overset{ess}{\subseteq} M$, in case for every non-zero submodule L of M, $K \cap L \neq 0$; equivalently for every submodule

$L \subseteq M$,

$$K \cap L = 0 \quad \text{implies} \quad L = 0.$$

Dually, a submodule K of M is superfluous (or small) in M, abbreviated

$$K \ll M,$$

in case for every submodule L of M,

$$K + L = M \quad \text{implies} \quad L = M.$$

Example 1. Every non-trivial submodule of $\mathbf{Z}_{p^\infty}$ is both essential and small as a Z-module.

Proposition 1.1.2. Let M be a module with submodules $K \subseteq N \subseteq M$ and $H \subseteq M$. Then

(i) $K \overset{ess}{\subseteq} M$ if and only if $K \overset{ess}{\subseteq} N$ and $N \overset{ess}{\subseteq} M$.

(ii) $H \cap K \overset{ess}{\subseteq} M$ if and only if $H \overset{ess}{\subseteq} M$ and $K \overset{ess}{\subseteq} M$.

Proof. (i) Let $K \overset{ess}{\subseteq} M$ and suppose $0 \neq L \subseteq M$, then $L \cap K \neq 0$. In particular this is true if $L \subseteq N$, so $K \overset{ess}{\subseteq} N$. Since $0 \neq L \cap K \subset L \cap N$ so $L \cap N \neq 0$ and hence $N \overset{ess}{\subseteq} M$. Conversely, suppose $K \overset{ess}{\subseteq} N \overset{ess}{\subseteq} M$ and $0 \neq L \subseteq M$, then $L \cap N \neq 0$ and hence $K \cap L \cap N \neq 0$, since $K \overset{ess}{\subseteq} N$. Thus $K \cap L \neq 0$.

(ii) One implication follows at once from (i). For the other, suppose $H \overset{ess}{\subseteq} M$ and $K \overset{ess}{\subseteq} M$. If $L \subseteq M$ and $L \cap H \cap K = 0$, then $L \cap H = 0$ because $K \overset{ess}{\subseteq} M$. Whence $L = 0$ because $H \overset{ess}{\subseteq} M$.

Lemma 1.1.3. A submodule ${}_RK \subseteq {}_RM$ is essential in M if and only if for each $0 \neq x \in {}_RM$ there exists an $r \in R$ such that $0 \neq rx \in K$.

Proof. ($\Longrightarrow$) If $K \overset{ess}{\subseteq} M$ and $0 \neq x \in M$, then $Rx \cap K \neq 0$.

($\Longleftarrow$) If the condition holds and $0 \neq x \in L \subseteq M$, then there is an $r \in R$ such that $0 \neq rx \in K \cap L$. Thus $K \overset{ess}{\subseteq} M$.

Proposition 1.1.4. Suppose that $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$, then:

(i) $K_1 \oplus K_2 \overset{ess}{\subseteq} M_1 \oplus M_2$ if and only if $K_1 \overset{ess}{\subseteq} M_1$ and $K_2 \overset{ess}{\subseteq} M_2$.

(ii) $K_1 \oplus K_2 \ll M_1 \oplus M_2$ if and only if $K_1 \ll M_1$ and $K_2 \ll M_2$.

Proof. See [1, Proposition 5.20].

Proposition 1.1.5. Let K, M and N be R-modules. If $K \overset{ess}{\subseteq} M$, then $K \oplus N \overset{ess}{\subseteq} M \oplus N$.

Proof. Let $m+n \in M \oplus N$. Since $K \overset{ess}{\subseteq} M$ by Lemma 1.1.3 there exists an $r \in R$ such that $0 \neq rm \in K$. Thus $0 \neq rm + rn = r(m+n) \in K \oplus N$ and hence $K \oplus N \overset{ess}{\subseteq} M \oplus N$ by Lemma 1.1.3.

Proposition 1.1.6. Let R be a ring and $a \in R$. If K is a small left ideal of R, then $Ka \ll Ra$.

Proof. Suppose $Ka + L = Ra$ for some left ideal L of Ra. Let $L' = \{r \in R | \ ra \in L\}$. Then L' is a left ideal of R and $L'a \subseteq L$. $L = Ra - Ka = (R - K)a$. Thus $R - K \subseteq L'$ and so $R = K + L'$.

Since $K \ll R$, $L' = R$ and hence $L = L'a = Ra$. Hence $Ka \ll Ra$.

Let M be a module and $K \subseteq N \subseteq M$. N is called an essential extension of K in M if $K \overset{ess}{\subseteq} N$.

Definition 1.1.7. A submodule N of a module M is called closed if

N has no proper essential extension in M.

Remark. By Zorn's Lemma each submodule of the module M is contained essentially in a closed submodule of M.

Definition 1.1.8. If M is a left R-module, then

$$\begin{aligned} SocM &= \sum\{K \subseteq M \mid K \text{is minimal in } M\} \\ &= \cap\{L \subseteq M \mid L \text{ is essential in } M\} \end{aligned}$$

Proposition 1.1.9. Let M be a module and $K \subseteq M$. Then $SocK = K \cap SocM \subseteq SocM$. In particular,

$$Soc(SocM) = SocM.$$

Proof. See [1, Corollary 9.9].

Corollary 1.1.10. Let M be a left R-module. Then $SocM \overset{ess}{\subseteq} M$ if and only if every non-zero submodule of M contains a minimal submodule.

Proof. $SocM \overset{ess}{\subseteq} M$ if and only if $SocM \cap L \neq 0$ for every non-zero submodule L of M. This implies if and only if $SocL \neq 0$, by Proposition 1.1.9. And $SocL \neq 0$ if and only if L contains a minimal submodule.

Let N be a submodule of M, N is homogeneous (or homogeneous component of $SocM$) in case for every simple submodule K of N, if $K \cong K' \subseteq M$, then $K' \subseteq N$.

Definition 1.1.11. If M is a left R-module, then

$$\begin{aligned} RadM &= \cap\{K \subset M \mid K \text{ is maximal in } M\} \\ &= \sum\{L \subset M \mid L \text{ is small in } M\} \end{aligned}$$

We will denote the Jacobson radical of a ring R by $J = J(R) = Rad_R R$.

Proposition 1.1.12. If $(M_\alpha)_{\alpha \in A}$ is an indexed set of submodules of M with $M = \oplus_A M_\alpha$, then $SocM = \oplus_A SocM_\alpha$ and $RadM = \oplus_A RadM_\alpha$.

Proof. See [1, Proposition 9.19].

Corollary 1.1.13. If M is semisimple, then $RadM = 0$.

Proof. Since M is semisimple we can write $M = \oplus_A P_\alpha$ where every P_α is simple, and since $RadP_\alpha = 0$ we have $RadM = \oplus_A RadP_\alpha = 0$.

An element $x \in R$ is left (right) quasi-regular in case $1 - x$ has a left (right) inverse in R.

Proposition 1.1.14. Let $J = J(R)$ be the Jacobson radical of R. Then $x \in J$ if and only if x is left (or right) quasi-regular.

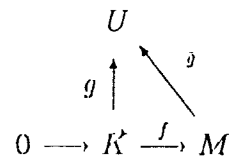
Proof. See [1, Theorem 15.3].

Proposition 1.1.15. If I is a left ideal of R and $Rad(\frac{R}{I}) = 0$, then $J(R) \subseteq I$.

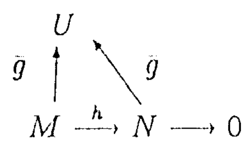
Proof. If $x \notin I$ then $x + I = \bar{x} \neq 0$, so $\bar{x} \notin Rad(\frac{R}{I})$. Hence there exists a maximal left ideal M of R with $I \subseteq M$ and $\bar{x} \notin \frac{M}{I}$. So $x \notin M$, and hence $x \notin J(R)$. Whence $J \subset I$.

1.2. Projective and Injective Modules

Definition 1.2.1. Let ${}_RU$ and ${}_RK$ be modules. If ${}_RM$ is a module, then U is M-injective (or U is injective relative to M) in case for each monomorphism $f : K \longrightarrow M$ and homomorphism $g : K \longrightarrow U$, there is an R-homomorphism $\bar{g} : M \longrightarrow U$ such that the following diagram commutes.



On the other hand, U is M-projective (or U is projective relative to M) in case for each epimorphism $h : M \longrightarrow N$ and each homomorphism $g : U \longrightarrow N$, for every left R-module N, there is an R-homomorphism $\bar{g} : U \longrightarrow M$ such that the following diagram commutes.



A module P is said to be projective in case it is M-projective for every module M. And a module E is injective in case it is M-injective for every module M.

Proposition 1.2.2. Let U and M be R-modules. U is M-injective if and only if every R-homomorphism from a submodule of M into U can be extended to M.

Proof. It is clear.

Lemma 1.2.3. [The Injective Test Lemma]. The following statements about a left R-module E are equivalent:

(i) E is injective;

(ii) E is injective relative to R;

(iii) For every left ideal $I \subseteq {}_RR$ and every R-homomorphism $h : I \longrightarrow E$ there exists an $x \in E$, such that h is right multiplication by x

$$h(a) = ax, \qquad \forall a \in I.$$

Proof. See [1, Lemma 18.3].

Examples. Q is an injective Z-module. On the other hand, Z is not an injective Z-module, since the homomorphism $f : 2\mathbf{Z} \longrightarrow \mathbf{Z}$ given by the rule $f(2n) = n$ can not be extended to a homomorphism from Z to Z.

Definition 1.2.4. An R-module F is free if it is isomorphic to a direct sum of copies of the left R-module R.

Proposition 1.2.5. A left R-module P is projective if and only if it is isomorphic to a direct summand of a free left R-module.

Proof. See [1, Proposition 17.2].

Lemma 1.2.6. Let M be a left R-module. If M is finitely generated and R-projective, then M is projective.

Proof. See [1, Corollary 16.14].

Proposition 1.2.7. P is projective if and only if every short exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ splits (i.e. $M \cong N \oplus P$).

Proof. See [1, Proposition 17.2].

Proposition 1.2.8. Every module is an epimorphic image of a projective module.

Proof. By Proposition 1.2.5 every free module is projective and every module is generated by ${}_R R$ so every module is an epimorphic image of a free module and so it is an epimorphic image of a projective module.

Proposition 1.2.9. Every left R-module can be embedded in an injective left R-module.

Proof. See [1, Proposition 18.6].

Proposition 1.2.10. Let $(M_\alpha)_{\alpha \in A}$ be an indexed set of left R-modules. Then

(i) $\oplus_A M_\alpha$ is projective if and only if each M_α is projective;

(ii) $\prod_A M_\alpha$ is injective if and only if each M_α is injective.

Proof. See [1, Corollary 16.11].

Recall that a pair (E, i) is called injective hull (or injective envelope) of M if E is an injective left R-module and

$$0 \longrightarrow M \xrightarrow{i} E$$

is an essential monomorphism ($\text{img } i \overset{ess}{\subseteq} E$).

Proposition 1.2.11. Every module M has the injective hull, which is unique minimal injective extension and maximal essential extension of M.

Proof. See [1, Proposition ...]

Recall that the injective hull of M is denoted by $E(M)$.

Proposition 1.2.12. In the category of left R-modules over a ring R.

(i) M is injective if and only if $M = E(M)$;

(ii) If $M \overset{ess}{\subseteq} N$, then $E(M) = E(N)$;

(iii) If $M \subseteq Q$, with Q injective, then $Q = E(M) \oplus E'$;

(iv) If $\oplus_A E(M_\alpha)$ is injective (for instance, if A is finite) then $E(\oplus_A M_\alpha) = \oplus_A E(M_\alpha)$.

Proof. Part (i) is immediate from the definition of the injective hull. For (ii) since $N \overset{ess}{\subseteq} E(N)$, if $M \overset{ess}{\subseteq} N$, then $M \overset{ess}{\subseteq} E(N)$ by Proposition 1.1.2 part (i). Since $E(M)$ is a maximal essential extension of M, $E(N) \subseteq E(M)$ and since $E(N)$ is injective and $E(M)$ is minimal injective extension of M, $E(M) \subseteq E(M)$. Thus $E(M) = E(N)$ for the proof of (iii) and (iv) see [1, Proposition 18.12].

Proposition 1.2.13. For a ring R the following are equivalent:

(i) Every direct sum of injective left R-modules is injective;

(ii) If $(M_\alpha)_{\alpha \in A}$ is an index set of left R-modules, then

$$E(\oplus_A M_\alpha) = \oplus_A E(M_\alpha)$$

(iii) R is a left noetherian ring.

Proof. See [1, Proposition 18.13]

Example. $E(Z) = Q$ and $E(Q) = Q$. Since Q is an injective Z-module and $Z \subseteq Q$ by Proposition 1.1.12 part (iii) $Q = E(Z) \oplus E'$. But Q is indecompossible and $E(Z) \neq 0$ so $E' = 0$ and $E(Z) = Q$.

Definition 1.2.14. A ring R is called left self-injective if ${}_RR$ is injective.

Proposition 1.2.15. For a left self-injective ring R the following codition are equivalent:

(i) left Noetherian;

(ii) right Noetherian;

(iii) left Artinian;

(iv) right Artinian.

In this case R is right self-injective.

Proof. See [5, p.182].

Examples. If $R = Z_6$, then every R-homomorphism from an ideal of R into R can be extended to an R-homomorphism from R into R. Thus by Proposition 1.2.2 and Lemma 1.2.3 R is left and right self-injective. And Z is not self-injective since it is noetherian but is not artinian.

Definition 1.2.16. A ring R is called right mininjective if every R-homomorphism from a minimal right ideal of R into R is given by left multiplication by an element of R; equivalently for each minimal right ideal K of R, every homomorphism $f : K \longrightarrow R$ extends to R.

Note that every self-injective ring is mininjective, and every polynomial ring $R[x]$ is mininjective, because it has no minimal ideal.

A ring R is called right (left) principally injective (or has right (left) principal extension property (P.E.P)) if every R-homomorphism from a principal right (left) ideal of R into R can be extended to an endomorphism of R.

Remark. Every principally injective ring is mininjective.