$*$-FRAMES AND G-FRAMES IN HILBERT C^*-MODULES

By

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To

my dear parents,

my kind husband

and

my lovely daughter; Sana

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Abstract

since 1950, frames have been a useful and important tool in signal processing, image processing, data compression, sampling theory, etc. In recent years, C^*-algebra theory and operator theory are being introduced to the study of frames and producing deep results in frame theory.

The main purpose of this dissertation is to study generalized frames and frames with vector valued bounds. Since the values of g-frames are operators, we are concerned with the composition of g-frames. We present the necessary and sufficient conditions for an operator to induce a g-frame consisting of a single term. Moreover, necessary and sufficient conditions are given for an operator to have the g-frame-preserving property. The dual of g-frames in Hilbert C^*-modules are also characterized.

Another purpose of the dissertation is to study a new version of frames in Hilbert C^*-modules. We introduce frames in Hilbert C^*-modules that have frame bounds in a C^*-algebra. Such a sequence is said to be a $*$-frame. We consider basic properties of $*$-frames. In some examples, the C^*-valued bounds and real-valued bounds for a sequence are compared. Also, we prove analogous results for $*$-frames and frames. We see that every frame in a Hilbert C^*-module is a $*$-frame and every $*$-frame has real-valued bounds. Then we obtain that $*$-frames are frames with different bounds. We also study the so-called "full Hilbert C^*-modules" whose properties may be interest to some readers.

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Introduction

Frames were first introduced in 1952 by Duffin-Schaeffer [17]. They abstracted the fundamental notion of Gabor [21] to study signal processing. It seems, however, that Duffin-Schaeffer ideas did not attract much interest outside the realm of nonharmonic Fourier series until the paper by Daubechies-Grassman-Mayer [16] was published in 1986.

The theory of frames has been rapidly generalized in form of the elements of frame and the space that the elements are chosen from it.

First, frames introduced in Hilbert spaces and studied in several parts; for example, frames of translate, Gabor frames (Weyl-Heisenberg frames), wavelet frames, frame multiresolution analysis, etc. [6, 7, 23, 24, 40]. Each of these classes is important and useful in a certain branch of science: Mathematic, dynamical systems, etc..

Also, frames in Hilbert spaces have been extended to frames in Banach spaces whose properties are specified accordingly [10, 15].

As the theory extends, the values of the frames have been changing. Generalized frames were originally introduced as sequences of bounded linear operators on Hilbert spaces [42]. Moreover, the elements of frames were changed as subspaces of Hilbert spaces and the definition of frame was presented for this sequence. This version of frame was called to be a fusion frame [12]. Properties of generalized frames and fusion frames have been considered by some researchers [35, 36, 43, 46].

It is well known that the theory of Hilbert C^*-modules has applications in the study of locally compact quantum groups, complete maps between C^*-algebras, non-commutative geometry, and KK-theory. There are many differences between Hilbert C^*-modules and Hilbert spaces. It is expected that problems about frames for Hilbert C^*-modules to be more complicated than those for Hilbert spaces. This makes the study of the frames for Hilbert C^*-modules important and interesting. So, frames in Hilbert spaces have been extended to frames in Hilbert C^*-modules [19]. In this case, the study of frames was not easily done because there were need to compare elements of C^*-algebra. However, an the equivalent definition presented in [25] paved the road for an easier access to frames in Hilbert C^*-modules than it was before. Frames, g-frames and fusion frames have been considered in Hilbert C^*-modules [4, 20, 22, 29, 30, 44].

As we saw in the above paragraphs, frames have undergone different extensions. Usually, the frame bounds were defined to be real-valued. Now, one may raise the following question: can one modify the definition such that the frame bounds to be objects other than nonnegative numbers?

In this thesis, we answer the question and introduce a version of frame in Hilbert C^*-modules with vector valued bounds. It means that we introduced a frame with frame bounds in a C^*-algebra. It is interesting that some elementary properties of frames are given for this sequence but the study of this type of frames is difficult because we use the relation between elements of a C^*-algebra and the equivalent definition [25] is no longer valid here.

The body of the thesis is divided into three chapters. The first chapter contains the basic tools that we need for the consequent chapters. The section 1.1 introduces some basic definitions of C^*-algebras. For studying frame theory in Hilbert C^*-modules, we need a partial ordering relation between elements of a C^*-algebra.To fulfill this, we recall the relation " $\leq$ " on a C^*-algebra and useful theorems such as Gelfand theorem, Spectral Mapping theorem, and Continuous Functional Calculus. The section also studies the tensor product of C^*-algebras that is also a C^*-algebra, and some properties of the new C^*-algebra.

In the second section of Chapter 1, some basic definitions and examples of Hilbert C^*-modules are introduced. It presents some identities and properties of such spaces. The definition of adjointable and bounded $\mathcal{A}$-module maps in Hilbert C^*-modules are given and further illustrated by certain examples. Also, we study certain properties of TT^* and T^*T associated to a given operator T, and find exact values for bounds of them with respect to $\mathcal{A}$-valued inner products. In the continuation, the tensor product of Hilbert C^*-modules and the tensor product of operators are explained. There exist many differences between Hilbert spaces and Hilbert C^*-modules. This section also states some differences that are needed. The relation between adjointable maps and bounded $\mathcal{A}$-module maps in Hilbert C^*-modules is compared with Hilbert case by a given example. Moreover in this section, we explained by an example that the Risze representation theorem is not valid in Hilbert C^*-modules. Types of Hilbert C^*-modules are introduced for example full and self-dual Hilbert C^*-modules. At the end of section, we present C^*-algebra $\mathcal{A}$ as a Hilbert $\mathcal{A}$-module and we get interesting results about it in the next parts.

Section 1.3 contains the definition of Bessel sequences, frames and some examples of them in Hilbert spaces. The operators corresponding to a given frame and properties of them are given. This section recalls the conditions that provides frame-preserving operator for a given operator, definition of dual frames and characterization of them.

In Section 1.4, we introduce frames in Hilbert C^*-modules. The theorems of Frank-Larson are presented that they explained for every finitely or countably generated Hilbert C^*-module there exists a frame and considered relations between frames in two Hilbert C^*-modules with different $\mathcal{A}$-valued inner products.

The g-Bessel sequences, g-frames and their duals, and g-orthonormal basis for Hilbert spaces are introduced in Section 1.5. Corresponding operators to a given g-frame are explained.

For more studying of g-frames in Hilbert spaces and Hilbert C^*-modules, we present Chapter 2 in three sections. Section 2.1 considers composition of the elements of two g-Bessel sequences and composition of the elements of two g-frames in Hilbert spaces. Also, we characterize g-frames by using g-orthonormal basis in Hilbert spaces.

The g-Bessel sequences, g-frames and their types, and g-Riesz bases for Hilbert C^*-modules are introduces in Section 2.2. In a proposition , the relation between frames and g-frames in Hilbert $\mathcal{A}$-module $\mathcal{A}$ is considered. We find a necessary and sufficient condition for an operator T such that $\{T\}$ is a g-frame. Also, the section recalls operators corresponding to a given g-frame and their properties. The relation between an adjointable operators θ from $\mathcal{H}$ into $\oplus_{j\in J}\mathcal{K}_j$ and g-Bessel sequences is obtained. Moreover, we modify the proof of Theorem 3.5 of [30] reveals that the bijectivity condition on T can be relaxed to the surjectivity of T and that the converse of the theorem remains true. At the end of Section 2.2, The family of g-frames is given associated to a g-frame by using its g-frame operator.

In the last section of Chapter 2, we characterize dual of g-frames and introduce types of them in Hilbert C^*-modules. The given results in this section are valid for g-frames in Hilbert spaces because of Hilbert spaces are Hilbert C^*-modules over C^*-algebra $\mathbb{C}$.

Chapter 3 presents a new version of frames in Hilbert C^*-modules. This chapter is organized in six sections. The frames with vector valued bounds, types of them and some examples are in this section. It compares $\mathcal{A}$-valued bounds and real valued bounds by examples. The $*$-frames in special Hilbert C^*-module $\mathcal{A}$ are determined.

Section 3.2 considers properties of operators associated to a given $*$-frame. And, by a theorem, it is shown that $*$-frames have real valued bounds and they can be studied as frame with different bounds.

In Section 3.3, $*$-frames on a special collection of Hilbert C^*-algebras are studied. When $\mathcal{A}$ is commutative, the given results in Section 3.1 and Section 3.2 are closed to ordinary frames. In the continuation of this section, some results about $*$-frames in Hilbert $\mathcal{A}$-module $\mathcal{A}$ and a Hilbert C^*-module $\mathcal{H}$ are given.

We give a $*$-frame for Hilbert $\mathcal{A}$-module $\mathcal{A}$ by a given $*$-frame in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ and then we obtain a result about full Hilbert $\mathcal{A}$-modules in Section 3.4. By an example, we show that the given condition in the result is necessary but is not sufficient. Moreover, $*$-frames for tensor product of Hilbert C^*-modules are constructed by $*$-frames in each Hilbert C^*-modules. At the end of Section 3.4, the necessary and sufficient conditions are given for operators that have $*$-frame-preserving property.

One of main results of this thesis is in Section 3.5. Frames in Hilbert $\mathcal{A}$-modules with different $\mathcal{A}$-valued inner products are studied by Frank-Larson [20]. Since our subject is to study of $*$-frames, we first extend the result [20] for $*$-frames in Section 3.5. Then $*$-frames in Hilbert C^*-modules with same vector space and different C^*-algebra are considered. Also, we obtain results about $*$-homomorphisms on C^*-algebras and the necessary and sufficient condition about a map that determines the relation between $*$-frames in Hilbert C^*-modules with different C^*-algebras. This section characterizes $*$-frames in a Hilbert $\mathcal{B}$-module with respect to a Hilbert $\mathcal{A}$-module. In the special case, $*$-frames in Hilbert $\mathcal{B}$-module $\mathcal{B}$ are characterized with respect to Hilbert $\mathcal{A}$-module $\mathcal{A}$.

In Section 3.6, we introduced dual $*$-frames and types of them. Also, the set of all duals associated to a given $*$-frame is characterized. These facts are valid for frames in Hilbert C^*-modules.

Chapter 1

Preliminaries

This chapter is intended to be as a memoir of some fundamental results in C^*-algebras, Hilbert C^*-modules and frame theory that will be used in the rest of the thesis. Consequently, we shall give no proofs as they can be found in all the standard textbooks of the subject. We will only prove some properties that are not present or are as exercises in text books.

1.1 C^*-algebras

The theory of C^*-algebras is an extension of the field of complex numbers. C^*-algebras are very well-behaved. A C^*-algebra is an algebra with an additional algebraic operation that this is called to be an involution. In this section, we study C^*-algebras to cover the basic results for studying Hilbert C^*-modules. And, it contains some properties of C^*-algebras that are necessary for studying the frame theory on Hilbert C^*-modules in the reminder of the thesis. For comprehensive accounts, we refer the interested readers to [5, 18, 26, 27, 34, 38, 41].

Let's begin with

Definition 1.1.1. A ***Banach algebra*** is an algebra $\mathcal{A}$ over the field of complex numbers equipped with a norm with respect to which it is a Banach space and which satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$. A Banach algebra $\mathcal{A}$ is called ***unital*** if it possesses a unit element or multiplicative identity. We denote the unit element of $\mathcal{A}$ by $1_{\mathcal{A}}$ or 1. A $*$***-algebra*** is a Banach algebra with an ***involution*** which is a map $a \longrightarrow a^*$ from $\mathcal{A}$ to $\mathcal{A}$ that satisfies

$$(a+b)^* = a^* + b^*, \qquad (\lambda a)^* = \overline{\lambda} a^*, \qquad (ab)^* = b^* a^*, \qquad a^{**} = a,$$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$. A Banach $*$-algebra that satisfies

$$\|a^* a\| = \|a\|^2, \qquad \forall a \in \mathcal{A},$$

is called a C^****-algebra***. A C^****-subalgebra*** of the C^*-algebra $\mathcal{A}$ is a vector space which is closed with respect to the multiplication and the involution of $\mathcal{A}$; every C^*-subalgebra is itself a C^*-algebra. For every subset S of $\mathcal{A}$, there is a smallest C^*-subalgebra of $\mathcal{A}$ containing S that is called the C^****-algebra generated by*** S.

The following example contains some of the C^*-algebras that we need.

Example 1.1.1. 1. *Let H be a Hilbert space. The set of all bounded linear operators on H is denoted by $L(H)$ which is a unital Banach algebra, with the operator norm, and the map $T \longrightarrow T^*$ is an involution that makes $L(H)$ into a C^*-algebra. (T^* is the adjoint of T.)*

2. *Let $l^\infty = l^\infty(\mathbb{N})$ be the space of all bounded complex-valued sequences that is a unital C^*-algebra with the following operations. For $u = \{u_i\}_{i\in\mathbb{N}}$ and $v = \{v_i\}_{i\in\mathbb{N}}$*

in l^∞, define

$$uv = \{u_i v_i\}_{i\in\mathbb{N}}, \qquad u^* = \{\overline{u_i}\}_{i\in\mathbb{N}}, \qquad \|u\| = \sup_{i\in\mathbb{N}} |u_i|.$$

Definition 1.1.2. Let $\mathcal{A}$ be a unital Banach algebra and let a be an element in $\mathcal{A}$. Then a is ***invertible*** if there is an element b in $\mathcal{A}$ such that $ab = ba = 1$. The set of all invertible elements in $\mathcal{A}$ is denoted by $Inv(\mathcal{A})$. If $a \in \mathcal{A}$, the ***spectrum*** of a is

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \ \ is \ \ not \ \ in \ \ Inv(\mathcal{A})\}.$$

Definition 1.1.3. Let $\mathcal{A}$ be a unital Banach algebra and $a, b \in \mathcal{A}$. We say that a ***commutes*** with b if $ab = ba$. ***The center*** of $\mathcal{A}$ is the set of all of elements of $\mathcal{A}$ such that commute with all of elements of $\mathcal{A}$; center of $\mathcal{A}$=$\{a \in \mathcal{A} : ab = ba, \ \ \forall b \in \mathcal{A}\}$. If the center of $\mathcal{A}$ is $\mathcal{A}$, then $\mathcal{A}$ is ***commutative***.

The Gelfand-Mazure theorem determines the collection of Banach algebras as follows.

Theorem 1.1.2. *[18] (**Gelfand-Mazure**) If $\mathcal{A}$ is a Banach algebra in which every nonzero element is invertible, then $\mathcal{A} \cong \mathbb{C}$.*

The following proposition includes some elementary facts about inverses and spectrum of elements in Banach $*$-algebras.

Proposition 1.1.3. *[18] Let $\mathcal{A}$ be a unital Banach $*$-algebra.*

i. $1^* = 1$.

ii. If a is invertible, then so is a^*, and $(a^*)^{-1} = (a^{-1})^*$.

iii. $\sigma(a^*) = \overline{\sigma(a)}$, for any $a \in \mathcal{A}$.

Definition 1.1.4. Let $\mathcal{A}$ be a unital Banach C^*-algebra and $a \in \mathcal{A}$.

1. a is ***self-adjoint*** or ***hermitian*** if $a = a^*$.

2. a is said to be ***normal*** if $aa^* = a^*a$.

3. a is ***positive*** if a is self-adjoint and $\sigma(a) \subseteq \mathbb{R}^+$. We write $a \geq 0$ to mean that a is positive.

4. a is ***strictly nonzero*** when a is nonzero and zero dosn't belong to $\sigma(a)$.

5. a is called to be ***strictly positive*** if a is strictly nonzero and positive.

The set of all of positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}^+$ and it is known that $\mathcal{A}^+$ is closed in $\mathcal{A}$ [34, P. 46]. Some properties and relations of positive elements of $\mathcal{A}$ are illustrated as follows: Using Theorem 1.1.4, we can extend definition of '$|.|$'. The ***absolute value*** of a is defined by $|a| := (a^*a)^{\frac{1}{2}}$ for $a \in \mathcal{A}$. For $a, b \in \mathcal{A}$, we write $a \leq b$ to mean $b - a \in \mathcal{A}^+$. The relation '$\leq$' is translation-invariant, that is $a + c \leq b + c$ for all $c \in \mathcal{A}$, and defines a partial ordering on $\mathcal{A}$.

Theorem 1.1.4. *[34] If a is an arbitrary element of a C^*-algebra $\mathcal{A}$, then a^*a is positive. Also, the set $\mathcal{A}^+$ is equal to $\{a^*a : a \in \mathcal{A}\}$.*

Proposition 1.1.5. *[41] Let $(a_k)_{k\in\mathbb{N}}$ be any sequence in a C^*-algebra $\mathcal{A}$. Then the sequence $(\sum_{k=1}^{n} a_k a_k^*)_{n\in\mathbb{N}}$ is an increasing sequence of positive elements which is strongly convergent in $\mathcal{A}''$ if $sup_n \| \sum_{k=1}^{n} a_k a_k^* \| < \infty$, ($\mathcal{A}''$ is the double dual of $\mathcal{A}$.). If the norm of the tail of $\sum_{k\in\mathbb{N}} a_k a_k^*$ tends to zero as $n \to \infty$, then $\sum_{k\in\mathbb{N}} a_k a_k^*$ is actually norm convergent in $\mathcal{A}$.*

Theorem 1.1.6. *[5] Let $\mathcal{A}$ be a unital Banach C^*-algebra and $a \in \mathcal{A}^+$. Then there exists a unique positive element $b \in \mathcal{A}^+$ such that $b^2 = a$. Moreover, b commutes with all the elements that commutes with a.*

The unique element b is called ***square root*** of a and is denoted by $\sqrt{a}$ or $a^{\frac{1}{2}}$.

Theorem 1.1.7. *[34] If a, b are positive elements in C^*-algebra $\mathcal{A}$, then the inequality $a \leq b$ implies that the inequality $a^{\frac{1}{2}} \leq b^{\frac{1}{2}}$.*

In this place, we can ask that is the converse of Theorem 1.1.7 true for every C^*-algebra? Answer this question is that if C^*-algebra $\mathcal{A}$ is commutative, then we have $0 \leq a \leq b$ implies that $a^2 \leq b^2$ on $\mathcal{A}$, [38, Proposition 1.3.9].

Proposition 1.1.8. *[34] Let $\mathcal{A}$ be a C^*-algebra.*

1. *If $a, b, c \in \mathcal{A}$ and $a \leq b$, then $cac^* \leq cbc^*$.*

2. *If $a, b \in \mathcal{A}$ and $0 \leq a \leq b$, then $\|a\| \leq \|b\|$.*

In the special case, if C^*-algebra $\mathcal{A}$ is unital, then the following relations are valid.

Proposition 1.1.9. *Let $\mathcal{A}$ be a unital C^*-algebra, and let a and b be two elements in $\mathcal{A}$. Then*

1. *$a \leq \|a\|$, [41].*

2. *If a and b are positive, then so is $a + b$. Moreover, ab is positive when $ab = ba$, [34].*

3. *If $\mathcal{A}$ is unital and a, b are positive invertible elements in $\mathcal{A}$, then $a \leq b$ concludes that $0 \leq b^{-1} \leq a^{-1}$, [34].*

4. *$a \leq b$ if and only if $-b \leq -a$, and $ta \leq tb$ for all $t \in \mathbb{R}^+$, [34].*

5. *If $0 \leq a \leq b$, then $a^t \leq b^t$ for all $t \in [0,1]$, [41].*

Definition 1.1.5. Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras.

1. A ***(Banach algebra) homomorphism*** from $\mathcal{A}$ to $\mathcal{B}$ is a bounded linear map $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ such that $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$ for all $a_1, a_2 \in \mathcal{A}$,

2. A ***$*$-homomorphism*** from $\mathcal{A}$ to $\mathcal{B}$ is a homomorphism φ such that preserves adjoints, that is, $\varphi(a^*) = \varphi(a)^*$ for all $a \in \mathcal{A}$. In addition, if φ is bijective, it is called to be a ***$*$-isomorphism***.

3. If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a linear map between C^*-algebras, it is said to be ***positive*** if $\varphi(\mathcal{A}^+) \subseteq \mathcal{B}^+$.

4. A ***multiplicative functional*** is a nonzero homomorphism from $\mathcal{A}$ into $\mathbb{C}$. The set of all multiplicative functionals on $\mathcal{A}$ is called ***the spectrum*** of $\mathcal{A}$ which is denoted by $\sigma(\mathcal{A})$.

Definition 1.1.6. Let a be an element in a C^*-algebra $\mathcal{A}$. The functional $\widehat{a}$ is defined by

$$\widehat{a}(\tau) = \tau(a), \qquad \forall \tau \in \sigma(\mathcal{A}).$$

We shall now completely determine the commutative C^*-algebras. The ***Gelfand*** theorem presents this result.

Theorem 1.1.10. *[34] (**Gelfand**) If $\mathcal{A}$ is a commutative unital C^*-algebra, then the* **Gelfand transform**

$$\Gamma : \mathcal{A} \longrightarrow \mathcal{C}(\sigma(\mathcal{A})), \qquad a \mapsto \widehat{a},$$

is an isomeric $*$*-isomorphism. Moreover,* $\sigma(a) = range(\widehat{a})$*, and* a *is invertible if and only if* $\widehat{a}$ *never vanishes.*

Proposition 1.1.11. *Let* $\mathcal{A}$ *be a commutative* C^**-algebra and* $a, b, c \in \mathcal{A}$*. If* c *is positive and* $a \leq b$*, then* $ca \leq cb$*. And if* c *is strictly positive and* $ca \leq cb$*, then* $a \leq b$*.*

In the proposition 1.1.11, the condition 'c commutes with a and b' can be instead of the commutative condition of $\mathcal{A}$.

Proof. The first part will obtain from Proposition 1.1.8 and the commutativity condition of $\mathcal{A}$. For the second part, assume that τ is an arbitrary element in $\sigma(\mathcal{A})$. By $ca \leq cb$, we have

$$0 \leq \tau(cb - ca) = \tau(c(b - a)) = \tau(c)\tau(b - a).$$

Since c is strictly positive, $\tau(c) > 0$. Then $\tau(b - a) \geq 0$, and $a \leq b$ by the Gelfand theorem. □

Proposition 1.1.12. *Let* $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ *be a* $*$*-homomorphism between unital* C^**-algebras.*

1. $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$*, [18].*

2. *For* $a \in \mathcal{A}$*, we have* $\sigma(\varphi(a)) \subseteq \sigma(a)$*, and if* φ *is injective, then* $\sigma(\varphi(a)) = \sigma(a)$*, [26].*

3. *The* $*$*-homomorphism* φ *is positive and increasing, that is,* $\varphi(\mathcal{A}^+) \subseteq \mathcal{B}^+$*, and if* $a_1 \leq a_2$*, then* $\varphi(a_1) \leq \varphi(a_2)$*, [34].*

4. *If* a *is invertible, then so is* $\varphi(a)$*, and* $\varphi(a^{-1}) = \varphi(a)^{-1}$*.*