بسم الله الرحمن الرحيم

IN THE NAME OF ALLAH

ON MINIMAX AND RELATED MODULES

BY

ZEINALABEDIN TAGIBEIGLOO

THESIS

SUBMITTED TO THE SCHOOL OF GRADUATE STUDIES IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF *MASTER OF SCIENCES (M.Sc.)*

IN

PURE MATHEMATICS

SHIRAZ UNIVERSITY

SHIRAZ, IRAN

EVALUATED AND APPROVED BY THE THESIS COMMITTEE AS:VERY GOOD

.... ERSHAD, M., Ph.D., ASSISTANT. OF PROF. MATH. (CHAIRMAN)

.... SHARIF, H., Ph.D., PROF. OF MATH.

.... HAKIM HASHEMI, M., Ph.D., ASSISTANT. PROF. OF MATH.

MARCH 2000

TO:

My Parents

ACKNOWLEDGMENT

I would like to express my thanks to Professor M. Ershad whose invaluable comments and suggestions made the completion of this thesis possible.

I am also grateful to Dr. H. Sharif and Dr. M. Hakim Hashemi who revised the text.

ABSTRACT

ON MINIMAX AND RELATED MODULES

BY

ZEINALABEDIN TAGIBEIGLOO

In this dissertation all rings are Noetherian and commutative with identity. An R-module M is called a minimax module if it has a finitely generated submodule U such taht $\frac{M}{U}$ is Artinian. In chapter 0 we will give some basic theorems and definitions and some concepts which will be needed later in our work.

In chapter 1 we introduce strongly faithful modules which one of its results is used in Chapter 2 to characterizae minimax modules.

In Chapter 2 we investegate minimax modules. And finally in chapter 3 we generelize minimax modules and we deduce the following theorem:

Let R be an arbitrary ring and M a radical R-module. Then

(i) M is locally a minimax module.

(ii) M is an extension of a coatomic module by a semi-Artinian, locally Artinian module.

TABLE OF CONTENTS

CHAPTER 0
BASIC DEFINITIONS AND RESULTS

Throughout this dissertation R will denote a commutative Noetherian ring with identity. We write $A \leq B (A < B)$ to mean that A is a submodule(proper submodule) of B.

Definition 0.1. A submodule K of M is essential(or large) in M, in case for every submodule L of M,

$$K \cap L = 0 \Rightarrow L = 0$$

Definition 0.2. An injective R-module which is an essential extension of M is called injective hull of M.

Definition 0.3. A submodule K of M is said to be small(or superfluous) in M, in case for every submodule L of M,

$$K + L = M \Rightarrow L = M.$$

If M is an R-module, we will denote by $Rad(M)$ and $E(M)$ the radical and the injective hull of M, respectively. We will write $M^{(k)}$ $(M^{(N)})$ to indicate a direct sum of k-copies(infinitely-copies) of M. The notations $A \leq_e B$ and $A << B$ will mean that A is an essential submodule of B and A is a small submodule of B, respectively.

If (R, m) is a local ring and M is an R-module, we denote by E the injective hull of R/m and set $M^0 = Hom_R(M, E)$. We define $Coass(M) = \{p \in spec(R) \mid p$ is the annihilator of an Artinian factor module of $M\}$ where $spec(R) = \{p \mid p \text{ is a prime ideal of } R\}$ and $Ass(M) = \{p \in spec(R) \mid p$ is the annihilator of an nonzero element $x \in M\}$.

Definition 0.4. A module M is called a minimax module, if it has a finitely generated submodule U such that M/U is Artinian.

Definition 0.5. A module M is called semi-Artinian if every proper submodule of M contains a minimax submodule.

Definition 0.6. A submodule U of an R-module M is called a coatomic submodule if every proper submodule of U is contained in a maximal submodule.

Definition 0.7. A nonzero module M is uniform provided that $I \cap K \neq 0$ for any two nonzero submodules I and K, or equivalently, every nonzero submodule is an essential submodule.

Definition 0.8. A monomorphism $f : K \to M$ is said to be an essential monomorphism in case $Imf \leq_e M$.

Definition 0.9. A module M that all of its factor modules are indecomposable is called a couniform module.

Definition 0.10. Let M be a module. M is an essential cover of a module N if there exists an small submodule U of M such that $M/U = N$.

Definition 0.11. Let $(T_\alpha)_{\alpha \in A}$ be an indexed set of simple submodules of M. If M is the direct sum of this set, then $M = \oplus_{\alpha \in A} T_\alpha$ is a semisimple decomposition of M. A module M is said to be semisimple in case it has a semisimple decomposition.

Definition 0.12. A set $\mathcal{M}$ of submodules of M is called coindependent if $U_i + (\cap_{j \neq i} U_j) = M$ for $U_i \in \mathcal{M}$.

Propsition 0.13. Let M be a module with submodules $K \leq N \leq M$ and $H \leq M$. Then

(1) $K \leq_e M$ if and only if $K \leq_e N$ and $N \leq_e M$.

(2) $H \cap K \leq_e M$ if and only if $H \leq_e M$ and $K \leq_e M$.

Proof. See [17, Proposition 5.16].

Proposition 0.14. If $K << M$ and $f : M \to N$ is a homomorphism then $f(K) << N$. In particular, if $K << M < N$ then $K << N$.

Proof. See [17, Lemma 5.18].

Lemma 0.15. For an R-module M and a prime ideal P of R the following statements are equivalent:

(1) $P \in Coass(M)$.

(2) $P = Ann_R(A)$ for an Artinian factor module A of M.

(3) $P \in Ass(Hom_R(M, C))$ for an Artinian R-module C.

If R is local, the above conditions are equivalent with

(4) $P \in Ass(M^0)$.

Proof. (1) $\Leftrightarrow$ (2) It is clear.

(2) $\Rightarrow$ (3) Let f be canonical epimorphism $f : M \to M/N = C$ then $P = Ann_R(f)$ therefore $P \in Ass((Hom_R(M, C))$ for an Artinian $R-$module C.

(3) $\Rightarrow$ (1) Let C be an Artinian $R-$module and $f \in Hom_R(M, C)$ such that $P = Ann_R(f)$ then $A = M/Kerf$ is an Artinian factor module of

M with $P = Ann_R(A)$. Hence $P \in Coass(M)$.

(4) $\Rightarrow$ (3) Put $C = E$.

(3) $\Rightarrow$ (4) Now $P \in Ass(Hom_R(M, C))$ and $C \subset E^n$ implies $P \in Ass(Hom_R(M, E)^n) = Ass(M^0)$ therefore $P \in Ass(M^0)$.

Corollary 0.16. If (R, m) is a local ring and M is an R-module then $Coass(M) = Ass(M^0)$.

Proof. It is clear.

Theorem 0.17.(Chevalley theorem) Let R be a complete semi-local ring, m the intersection of its maximal ideals, and (a_n) descending sequence of ideals of R such that $\cap_{n=0}^{\infty} a_n = (0)$. Then there exists an integral valued function $s(n)$ which tends to infinitely with n, such that $a_n \subset m^{s(n)}$.

Proof. See [11, VIII, Sec.5, Theorem 13].

Theorem 0.18. Let R be a commutative Noetherian ring and let M be a finitely generated R-module. Then every set of submodules of M which is totally ordered by inclusion is countable.

proof. See [1, Theorem 1.1].

Definition 0.19. A right Ore domain is a (nonzero) integral domain R such that any two nonzero elements of R have a nonzero common right multiple. Equivalently, the intersection of any two nonzero right ideals of R must be nonzero. (Of course, all commutative domains are Ore domain).

We write $ann(x)$ for the annihilator (in R) of any element $x \in M$, and we write $ann(M)$ for the annihilator of M.

Definition 0.20. The module M is said to be bounded if $ann(M)$ is nonzero; otherwise, M is called unbounded.

Definition 0.21. A right module M over a right Ore domain R is a torsion module provided that each element of M can be annihilated by a nonzero element of R.

Definition 0.22. Given any set I, a cofinite subset of I is any $J \subset I$ for which $I - J$ is finite.

Definition 0.23. A (right) Ore domain R is (right) productively bounded if for any nonempty family $(M_i)_{i \in I}$ of (right) R-modules such that the direct product $\prod_{i \in I} M_i$ is a torsion module, there is a cofinte subset J of I such that $\prod_{i \in J} M_i$ is bounded,i.e., $\prod_{i \in J} M_i$ can be annihilated by some nonzero element of R.

Corollary 0.24. Each right Ore domain, for which the right Krull dimension exists and is countable, is right productively bounded.

In particular, any commutative Noetherian domain with finite classical Krull dimension is productively bounded.

Proof. see [3, Corollary 5.6].

Let R be an arbitrary Noetherian ring. By Ω we denote the set of all maximal ideals of R. Let $a \subset R$. Then we write $M[a] = \{x \in M \mid ax = 0\}$.

An R-module M is called radical if it has no maximal submodules,i.e., $Rad(M) = M$. By $P(M)$ we denote the sum of the radical submodules of M. $P(M)$ is the largest radical submodule of M. If

$P(M) = 0$, M is called reduced.

For any R-module M we denote by $L(M)$ the sum of all Artinian submodules of M. $L(M)$ is the largest semi-Artinian submodule of M. $L(M)$ always has a decomposition $L(M) = \oplus_{m\in\Omega} L_m(M)$, where $L_m(M) = \sum_{n=1}^{\infty} M[m^n]$ is called the m-primary component of $L(M)$. If $L(M) = 0$, M is called socle-free.

Let M be an R-module. The Goldie-dimension of M(we write $dim(M)$) can be defined in the following way:
$dim(M) = n$ if and only if M has an essential submodule B that is a direct sum of n uniform modules; $dim(M) = \infty$ if and only if M contains a submodule which has an infinite decomposition.

Definition 0.25. A submodule K of a module M is said to be a complement submodule provided that there is a submodule S such that K is maximal in the set of all submodules T such that $S \cap T = 0$. In this case, K is said to be a complement of S.

Theorem 0.26. (Goldie-dimension Theorem) For an R-module M and its injective hull $E(M)$, the following are equivalent:
(1) M has ACC for direct summands, that is, every nonempty set of independent submodules of M is finite,i.e., M contains no infinite direct sum of nonzero submodules.
(2) $E(M)$ is a direct sum of finite number of indecomposable modules.
(3) M contains an essential submodule which is a direct sum of a finite number of uniform submodules.
(4) M has the ACC on complement submodules.

Proof. See [18, 1.12.A, Goldie-dimension Theorem].

Proposition 0.27. Let M be an R-module such that $L(M) = M$ and $Ass(M)$ is finite. Then $\cap Coass(M) = \sqrt{Ann_R(M)}$.

Proof. See [15, Anhang Satz].

Theorem 0.28.(Anhang theorem) Let M be an R-module. The following statements are equivalent:

(1) M is a minimax module.

(2) Every factor module of M has finite Goldie-dimension.

(3) In every ascending chain $U_1 \subset U_2 \subset \ldots$ of submodules of M almost all factors U_{i+1}/U_i are of finite Goldie-dimension.

(4) In every descending chain $U_1 \supset U_2 \supset \ldots$ of submodules of M almost all factors U_i/U_{i+1} are of finite Goldie-dimension.

Proof. See [15, Anhang Satz].

Proposition 0.29. If every proper submodule of M is contained in a maximal submodule of M, then $Rad(M)$ is the unique largest small submodule of M.

Proof. See [17, Proposition 9.18].

Definition 0.30. We call M weakly complemented if, for every submodule U of M, there is a submodule V of M such that $U + V = M$ and $U \cap V$ is small in M.

Let R be a ring and let $R = m_0 \supset m_1 \supset m_2 \supset \ldots$ be a sequence of ideals of R. We define the completion $\hat{R}$ of R with respect to the m_i to be the inverse limit of the factor ring R/m_i,i.e., $\hat{R} = lim_{\leftarrow} R/m_i =$

$\{g = (g_1, g_2, \ldots) \in \prod_i R/m_i \mid g_j \equiv g_i (\bmod m_i) \text{ for all } j \geq i\}$.

Theorem 0.31. For every R-module M the following statements are equivalent:

(1) Every coindependent set of submodules of M is finite.

(2) If $X_1, X_2, \ldots$ is a chain of submodules of M with $(\cap_{i=1}^{n-1} X_i) + X_n = M$ for all $n \geq 2$, then $X_i = M$ for all i.

(3) M is weakly complemented, and every factor module of M has ACC for direct summands.

(4) M is an essential cover of a finite direct sum of indecomposable modules.

(5) M is an essential cover of an Artinian module.

If R is local then the above conditions are equivalent with the following condition

(6) M^0 has finite Goldie-dimension as $\hat{R}$-module.

Proof. See [14, Satz 3.6].

Lemma 0.32. Suppose that U is a submodule of M. Then U is small in M, if U_m is small in M_m for all maximal ideals m of R.

Proof. See [12, lemma 4.1].

Theorem 0.33. For every R-module M the following are equivalent:

(1) M is coatomic.

(2) There is an integer $e \geq 1$, such that $\frac{M}{Ann_R(m^e)}$ is finitely generated.

(3) There is an integer $e \geq 1$, such that $m^e M$ is finitely generated.

Proof. see [12, Satz A].

Definition 0.34. If (R, m) is local we call an R-module M discrete if there is an $n \geq 1$ with $m^n M = 0$.

Corollary 0.35. An R-module M is coatomic if and only if M is the sum of a finitely generated and a discrete submodule.

Proof. It is clear.

Lemma 0.36. Suppose that m is a maximal ideal of R. If there is a coatomic submodule A of M which $(M/A)_m = 0$ then M_m is a coatomic R_m-module.

Proof. See [12, Lemma 3.2].

Lemma 0.37. If M_m is a coatomic R_m-module for all maximal ideal m of R then M is coatomic.

Proof. See [12, Folgerung Zu Lemma 1.1].

Proposition 0.38. Let M be a module over a Noetherian ring R. Then the condition $M \neq 0$ is equivalent to $Ass(M) \neq 0$.

Proof. See [2, IV, Sec.1.1, Corollary 1].

Lemma 0.39. If M is an extension of a coatomic module by a module N such that $L(N) = N$, then

(a) For all prime ideal $p \notin \Omega$, M_p is a finitely generated R_p-module.

(b) For all $p \in Ass(P(M))$, $dim(R/p) \leq 1$.

Proof. See [14, Lemma 1.1].

proposition 0.40. If M is a coatomic module, then $Ass(M)$ is finite.

Proof. See [12, Folgerung Zu Lemma 2.1].

Theorem 0.41. For every R-module M the following are equiva-