IN THE NAME OF GOD

HOMOLOGY DECOMPOSITIONS FOR CLASSIFYING SPACES OF COMPACT LIE GROUPS

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To my dear father, my
beloved mother
and my advisor
Dr. Mehdi Hakim-Hashemi

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ABSTRACT

HOMOLOGY DECOMPOSITIONS FOR CLASSIFYING SPACES OF COMPACT LIE GROUPS

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Let p be a fixed prime number and G be a compact Lie group. A homology decomposition for the classifying space BG is a way of building BG up to mod p homology as a homotopy colimit of classifying spaces of subgroups of G. In our paper we devlope techniques for constructing such homology decomposition. In [21], **Jackowski, MacClure and Oliver** construct a homology decomposition of BG by classifying spaces of p-stubborn subgroups of G. Their decomposition is based on the existence of a finite-dimentional mod p acyclic G-CW-complex with restricted set of orbit types. We apply our techniques to give a parallel proof of the p-stubborn decomposition of BG which does not use this geometric construction. For our purpose, in chapter 1 and chapter 2 we give some basic definitions. In chapter 3, we devlope homotopy colimit interpretations of certain known constructions, such as the Borel construction. We also discuss some basic properties of these constructions. In chapter 4, we use the fact that a collection of subgroups of G, say $\mathcal{C}$, is ample if the G-space $E\mathcal{O}(\mathcal{C})$ has mod p homology of a point, and state some conditions that

imply mod p acyclicity of this space. In this chapter we give theorems that will allow us to reduce the proof of the homology decomposition theorem to the case of compact Lie groups with simpler structures (e.g., groups of smaller dimension or groups with π_0 a p-group). In chapter 5, we use the fact that the E_2-term of the standard Bousfield-Kan cohomology spectral sequence associated to $\text{hocolim}_{\mathcal{O}(\mathcal{C})} EG \times_G F$ can be identified with the ordinary equivariant cohomology groups of the G-space $E\mathcal{O}(\mathcal{C})$ with certain coefficient systems. In fact a collection $\mathcal{C}$ is sharp if and only if $E\mathcal{O}(\mathcal{C})$ has equivariant cohomology of a point and define the transfer map on equivariant cohomology which is used to show that for suitable collections $\mathcal{C}$ the E_2-term of the spectral sequence which introduced above, has appropriate vanishing properties. In chapter 6, we establish a relation between the p-toral and p-stubborn decompositions and finally in chapter 7, we complete our proof of the homology decomposition theorem, i.e., we prove that:

Theorem: Suppose that G is a compact Lie group that contains an element of order p. Then the collection of nontrivial p-toral subgroups of G is ample and the collection of nontrivial p-stubborn subgroups of G is sharp.

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CHAPTER 1

INTRODUCTION

Before giving the introduction we need some basic definitions.

DEFINITION 1.0.1. *Let C be a category for which $ob(C)$ is a set, then we say that C is a* **small category**.

DEFINITION 1.0.2. *A* **topological category** *$\mathcal{D}$ is a category $\mathcal{D}$ with topologized morphism sets such that composition is continuous and for each object d of $\mathcal{D}$ the inclusion $id_d \hookrightarrow Mor(d,d)$ is a closed cofibration.*

DEFINITION 1.0.3. *A* **topological group** *is a Hausdorff space G together with a continuous multiplication $G \times G \to G$ (usually denoted by juxtaposition $(g,h) \mapsto gh$), which makes G into a group and such that the map $g \mapsto g^{-1}$ of $G \to G$ is continuous.*

DEFINITION 1.0.4. *The* **identity component** *of a topological group G is a path component including the identity of G.*

DEFINITION 1.0.5. *A* **compact Lie group** *is a topological group G where G is a compact manifold and the maps $(g,h) \mapsto gh$ and $g \mapsto g^{-1}$ are in C^∞.*

DEFINITION 1.0.6. *A* **torus** *is a compact Lie group which is isomorphic to $\mathbb{R}^n/\mathbb{Z}^n = (\mathbb{R}/\mathbb{Z})^n$ for some $n \geq 0$. The number n is called the* **rank** *of torus.*

EXAMPLE 1.0.1. *Consider the covering map*

$$R \xrightarrow{\pi} S^1 \text{ given by}$$

$$x \mapsto e^{2\pi i x}$$

this map is a defeomorphism and $ker\pi = \mathbb{Z}$*. Thus* $\mathbb{R}/\mathbb{Z} \simeq S^1$ *and so* $T^2 = S^1 \times S^1 = (\mathbb{R}/\mathbb{Z})^2$.

DEFINITION 1.0.7. *A* **topological transformation group** *is a triple* (G, X, θ)*, where* G *is a topological group,* X *is a Hausdorff topological space and* $\theta : G \times X \to X$ *is a map such that*

1. $\theta(g, \theta(h, x)) = \theta(gh, x)$ *for all* $g, h \in G$
2. $\theta(e, x) = x$ *for all* $x \in X$*, where e is the identity of* G.

The map θ *is called an action of* G *on* X*. The space* X *together with a given action* θ *of* G *is called* G*-space.*

DEFINITION 1.0.8. *Let* I *be a small category, an* I*-diagram in a category* C *is a functor*

$$F : I \to C$$

$$x \longmapsto F(x).$$

EXAMPLE 1.0.2. *The functor*

$$F : \{1, 2, 3, 1 \to 2, 1 \to 3\} \to Top \text{ given by}$$

$$F(1) = X \quad F(2) = Y \quad F(3) = Z$$

gives the following I*-diagram:*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \\ Z & & \end{array}$$

EXAMPLE 1.0.3. *Let G be a group. Let $\mathcal{G}$ be a category with G as the only object and $Mor(G,G)$ being the left translations, i.e.,*

$$Mor(G,G) = \{L_g : G \to G,\ L_g(h) = gh\ \}.$$

Then $\mathcal{G}$ is a small topological category. Any $\mathcal{G}$-diagram over Top is a G-space. Let $X : \mathcal{G} \to Top$ be a G-space with $X(G) = X$, then $hocolim_{\mathcal{G}} X$ is called the **Borel construction**. *It is also denoted by $EG{\times}_G X$. This makes $EG{\times}_G -$ into a functor from Top to Top.*

Let $X = *$, the one point space. Then $hocolim_{\mathcal{G}} * (= EG{\times}_G *)$ is called the **classifying space** *of G and is also denoted by BG. This space is in fact the classifying space of the group G which appears in the context of fiber bundles with structure group G (see* **Housemoller** [18]*).*

DEFINITION 1.0.9. *Let G be a topological group. A set C of closed subgroups of G is called a* **collection** *if it is closed under conjugation in G. Let C be a collection, the C-orbit category $\mathcal{O}(C)$ is the category with objects being the G-sets G/H for H in C and morphisms are G-maps. Let $J_{\mathcal{O}(C)}$ (or simply J) be the inclusion functor*

$$J : \mathcal{O}(C) \to G\text{-}Top$$

where G-Top is the category of G-spaces and G-maps.

Let p be a fixed prime number and G be a compact Lie group. A **homology decomposition** for the classifying space BG is a mod p homology isomorphism

$$\text{hocolim}_{\mathcal{D}} F \xrightarrow{p} BG$$

where $\mathcal{D}$ is a small category, F is a functor from $\mathcal{D}$ to the category of spaces and for each object d of $\mathcal{D}$, $F(d)$ has the homotopy type of BH for some subgroup H of G. In [19] **Jackowski, McClure** and **Oliver** construct a homology decomposition of BG by classifying spaces of p-stubborn subgroups of G. Their decomposition is based on the existence of a finite dimensional mod p acyclic G-CW-complex with resstericted set of orbit types.

In **[30] Strounine** regards $\mathcal{O}(\mathcal{C})$ as a topological category, where the set of objects is discrete and the morphisms set has compact-open topology.

NOTE 1. *Let $\mathcal{O}(G)$ be the orbit category and $J : \mathcal{O}(G) \to G\text{-}Top$ be the inclusion functor. The Borel construction on a space X is also a G-space, one has $EG\times_G - : G\text{-}Top \to G\text{-}Top$ as a functor. Now let $EG\times_G J$ be the composite*

$$\mathcal{O}(G) \xrightarrow{J} G\text{-}Top \xrightarrow{EG\times_G -} G\text{-}Top.$$

Since the Borel construction is a homotopy colimit, it commutes with another homotopy colimit. Precisely, let $\mathcal{C} \subset \mathcal{O}(G)$ be the full subcategory. Then $hocolim_{\mathcal{O}(\mathcal{C})} J \in ob(G\text{-}Top)$. So one has

$$EG\times_G(hocolim_{\mathcal{O}(\mathcal{C})} J) \simeq hocolim_{\mathcal{O}(\mathcal{C})}(EG\times_G J).$$

But for each $G/H \in ob(\mathcal{O}(G))$ one has (see **Segal** [28]*)*

$$(EG\times_G J)(G/H) = EG\times_G G/H \equiv EH/H \cong BH.$$

So

$$EG\times_G hocolim_{\mathcal{O}(\mathcal{C})} J \cong hocolim_{\mathcal{O}(\mathcal{C})}(EG\times_G J) \cong hocolim_{\mathcal{O}(\mathcal{C})}(EG/H).$$

Let $G/G = *$ be the one point space with trivial action of G. Then the maps $G/H \to G/G = *$ induce natural maps

$$EG\times_G G/H \to EG\times_G * = BG.$$

Now by the universal property of homotopy colimit there is a map

$$hocolim_{\mathcal{O}(C)}(EG\times_G G/H) \to BG.$$

DEFINITION 1.0.10. *A collection of subgroups C is* ample *if the map*

$$hocolim_{\mathcal{O}(C)}EG\times_G J \to BG$$

has mod p homology of a point.

The Borel construction is itself a homotopy colimit and hence commutes with homotopy colimits. This gives

$$EG\times_G \text{hocolim}_{\mathcal{O}(\mathcal{C})}J \cong \text{hocolim}_{\mathcal{O}(\mathcal{C})}EG\times_G J.$$

To prove that a collection $\mathcal{C}$ is ample, it suffices to show that in the fibration

$$\text{hocolim}_{\mathcal{O}(\mathcal{C})}J \to EG\times_G \text{hocolim}_{\mathcal{O}(\mathcal{C})}J \to BG$$

the fiber $\text{hocolim}_{\mathcal{O}(\mathcal{C})}J$ is $\mathbb{F}_p$-acyclic (i.e., the map

$$\text{hocolim}_{\mathcal{O}(\mathcal{C})}J \to *$$

induces isomorphism on mod p homology).

Suppose that $\mathcal{O}(\mathcal{C})$, for some $\mathcal{C}$, has finite morphisms set. Then the standard **Bousfield-Kan** cohomology spectral sequence of a homotopy colimit (see [4]) associated to

$$\text{hocolim}_{\mathcal{O}(\mathcal{C})}EG\times_G J$$

has the form:

$$E_2^{i,j} = lim^i_{\mathcal{O}(\mathcal{C})}H^j(EG\times_G J, \mathbb{F}_p) \tag{1}$$

where $\lim^i$ is the i-th derived functor of the inverse limit functor on the category of groups.

DEFINITION 1.0.11. *An ample collection C of subgroups of G is said to be* **sharp** *if $\mathcal{O}(C)$ is a discrete category (in the sense that the morphisms set of $\mathcal{O}(C)$ has discrete topology) and the spectral sequence* (1) *collapses, that is $E_2^{i,j} = 0$ for $i > 0$ and $E_2^{0,j} \approx H^j(BG, \mathbb{F}_p)$.*

In order to state the main theorem we need:

DEFINITION 1.0.12. *A compact Lie group P is called* **p-toral** *if its identity component P_0 is a torus and P/P_0 is a finite p-group.*

DEFINITION 1.0.13. *A p-toral subgroup P of G is called* **p-stubborn** *if $N_G(P)/P$ is finite and has no nontrivial normal p-subgroup.*

THEOREM 1.0.1. *Suppose that G is a compact Lie group that contains an element of order p. Then the collection of nontrivial p-toral subgroups of G is ample and the collection of nontrivial p-stubborn subgroups of G is sharp.*

In this thesis p is a fixed prime and $\mathbb{F}_p$ is the field with p elements.

CHAPTER 2

BASIC DEFINITIONS

2.1. Simplicial sets

DEFINITION 2.1.1. *A* **simplicial object** X *over a category* C *consists of*

1. *For every integer* $n \geq 0$ *an object* $X_n \in C$ *and*
2. *For every pair* (i, n) *with* $0 \leq i \leq n$, *face and degeneracy maps* $X_n \xrightarrow{d_i} X_{n-1}$ *and* $X_n \xrightarrow{s_i} X_{n+1}$ *satisfying the simplicial identities:*

$$d_i d_j = d_{j-1} d_i \text{ if } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ 1 & \text{if } i = j, j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases}$$

$$s_i s_j = s_j s_{i-1} \text{ if } i > j$$

similarly, a **simplicial map** $f : X \to Y$ *between two simplicial objects consists of maps* $f_n : X_n \to Y_n \in C$ *which commute with face and degeneracy maps, i.e.,* $d_i f_n = f_{n-1} d_i$ *and* $s_i f_n = f_{n+1} s_i$ $\forall\, i, n$.

DEFINITION 2.1.2. *A simplicial object over the category of sets is called a* **simplicial set** *and we denote the category of simplicial sets by* $\mathcal{S}$.

For $X \in \mathcal{S}$, the elements of X_n are called n-simplices; 0-simplices are sometimes called vertices.

DEFINITION 2.1.3. *For $X \in \mathcal{S}$, a simplex $x \in X$ is called* **degenerate** *if $x = s_i x'$ for some $x' \in X$ and otherwise it is called* **non-degenerate**.

DEFINITION 2.1.4. *For $X \in \mathcal{S}$, the* **n-skeleton** *$X_n \in \mathcal{S}$ is the subobject generated by all simplices of X of dimensions less than or equal to n.*

One can get a better idea what, in general, a simplicial set looks like by considering the singular realization functors between the category $\mathcal{S}$ of simplicial sets and the category *Top* of topological spaces. To define these we need:

DEFINITION 2.1.5. *For every $n \geq 0$, the* **topological n-simplex**, *$\underline{\Delta}_{[n]}$, is the subspace of $n+1$-dimensional Euclidean space consisting of the points $(t_0, \dots, t_n)$ for which $\sum_{i=0}^{n} t_i = 1$ where $0 \leq t_i \leq n$, $\forall i$. Similarly, for all $0 \leq i \leq n$, the standard maps $\underline{\Delta}_{[n-1]} \xrightarrow{\underline{d}^i} \underline{\Delta}_{[n]}$ and $\underline{\Delta}_{[n+1]} \xrightarrow{\underline{s}^i} \underline{\Delta}_{[n]}$ are given by the formulas*

$$\underline{d}^i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

$$\underline{s}^i(t_0, \dots, t_{n+1}) = (t_0, \dots, t_i + t_{i+1}, \dots, t_{n+1})$$

that satisfy the dual of the simplicial identities, i.e.,

$$\underline{d}^j \underline{d}^i = \underline{d}^i \underline{d}^{j-1} \text{ if } i < j$$

$$\underline{s}^j \underline{d}^i = \begin{cases} \underline{d}^i \underline{s}^{j-1} & \text{if } i < j \\ 1 & \text{if } i = j, j+1 \\ \underline{d}^{i-1} \underline{s}^j & \text{if } i > j+1 \end{cases}$$

$$\underline{s}^j \underline{s}^i = \underline{s}^{i-1} \underline{s}^j \text{ if } i > j.$$