To my family

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ABSTRACT

CYCLICITY OF SOME OPERATORS ON SPACES OF ANALYTIC FUNCTIONS

BY

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In this thesis, an n-tuple of operators is a finite sequence of length n of commuting continuous linear operators $T_1, T_2, ..., T_n$ acting on a locally convex topological space X. An n-tuple $(T_1, T_2, ..., T_n)$ is said to be hypercyclic, if there exists a vector $x \in X$ such that the set $\{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n}x : k_i \geq 0, i = 1, 2, ...n\}$ is dense in X. If there exists a vector $x \in X$ such that the set $\{\lambda {T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n}x : \lambda \in \mathbb{C}, k_i \geq 0, i = 1, 2, ...n\}$ is dense in X, then $(T_1, T_2, ..., T_n)$ is said to be a supercyclic n-tuple of operators.

In this thesis, in the first part, we give sufficient conditions under which the adjoint of an n-tuple of a weighted composition operator on a Hilbert space of analytic functions is hypercyclic. In the second part, we show that if T is a supercyclic ℓ-tuple of $n \times n$ complex matrices, then $\ell \geq n$. We also prove that there exists a supercyclic n-tuple of diagonal $n \times n$ matrices. Furthermore, if $T = (T_1, ..., T_n)$ is a supercyclic n-tuple of $n \times n$ complex matrices, then T_j's are simultaneously diagonalizable.

Contents

Chapter 1

Introduction

1 Introduction

If T is a bounded linear operator on a Banach space X, i.e. $T \in B(X)$, then the orbit of a vector $x \in X$ for T is the set $Orb(T, x) = \{T^n x : n \in \mathbb{N} \cup \{0\}\}$. A vector x is called hypercyclic vector for T if $Orb(T, x)$ is dense in X or, in other words, there is no proper closed T-invariant subset of X containing x. T is called hypercyclic if it has a hypercyclic vector. A vector $x \in X$ is said to be cyclic vector for an operator $T \in B(X)$ if the linear span of $Orb(T, x)$ is dense in X. An operator $T \in B(X)$ is cyclic if it has a cyclic vector. Furthermore, a vector $x \in X$ is called supercyclic vector for an operator $T \in B(X)$ if $\mathbb{C}.Orb(T, x) = \{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N} \cup \{0\}\}$ is dense in X. Finally an operator $T \in B(X)$ is supercyclic if it has a supercyclic vector. It is evident that hypercyclicity implies cyclicity and supercyclicity.

The first example of hypercyclicity appeared in the space of entire functions in 1929 by Birkhoff [4]. He showed essentially that the translation operator is a hypercyclic operator, while in 1952, MacLane [28] proved the hypercyclicity of the differentiation operator. Hypercyclicity on Banach spaces was discussed in 1969 by Rolewicz [29], who showed that whenever $|\lambda| > 1$, λT is hypercyclic where T is the unilateral backward shift on ℓ^p $(1 \leq p \leq \infty)$ or c_0.

In 1982, C. Kitai in her PH.D. Dissertation [27], determined the conditions that ensure a continuous linear operator to be hypercyclic. This result, commonly referred to as the Hypercyclicity Criterion, was never published, and a few years later it was rediscovered in a broader form by R. M. Gethner and J. H. Shapiro [19], who used it to unify the previously mentioned results of Birkhoff, MacLane and Rolewicz, among others.

Hypercyclic tuples of operators were introduced by Kerchy in [26] and Feldman in [14, 15]. The hypercyclicity of tuples of the adjoint of the weighted composition operators were investigated in [36].

Adjoints of multiplication operators contain hypercyclic operators [20]. For other useful references one can see [13, 17, 18].

The hypercyclicity of composition operators also has been considered by Bourdon and Shapiro in [5, 6]. They studied the hypercyclicity of composition operators on Hardy space H^2. Good sources of background information on composition operators include [8, 10, 33].

Also, Shapiro gave a complete characterization of hypercyclic composition operators on $H(\mathbb{D})$ in [34].

In [38] the authors showed that weighted composition operators with non-constant weight functions can be hypercyclic on $H(\mathbb{D})$.

A nice condition for hypercyclicity is the Hypercyclicity Criterion which was developed independently by Kitai [27] and Gethner and Shapiro [19]. This criterion has been used to show that certain classes of many operators are hypercyclic. Some reformulation of this criterion is given in [39]. In [40] it is shown that hypercyclicity can occur for the adjoint of a weighted composition operator. In [24] it is given sufficient conditions under which a weighted composition operator on a Hilbert space of analytic functions is not weakly supercyclic. Hypercyclicity, supercyclicity and cyclicity of the adjoint of a weighted composition operator on a Hilbert space of analytic functions were discussed by Kamali, Khani Robati and Hedayatian in [25].

The word ”hypercyclic” comes from the much other notion of a cyclic operator. An operator $T \in B(X)$ is said to be cyclic if there exists a vector $x \in X$ such that the linear span of $Orb(T, x)$ is dense in X. This notion is of course related to the famous invariant subspace problem: given an operator $T \in B(X)$, is it possible to find a non-trivial closed subspace $F \subset X$ which is T-invariant (i.e. $T(F) \subset F$)? Here, non-trivial means that $F \neq \{0\}$ and $F \neq X$. Clearly, for all $x \in X$, the closed linear span of $Orb(T, x)$ is an invariant subspace for T; hence, T has not non-trivial invariant closed subspace iff every non-zero vector $x \in X$ is a cyclic vector for T.

In this chapter we state the basic definitions and notations which are used in other chapters.

1.1 Analytic functions spaces

Definition 1.1.1. ([10]) A Hilbert space of complex valued functions on a set X is called a functional Hilbert space on X if the vector operations are the pointwise operations, $f(x) = g(x)$ for each x in X implies $f = g$, $f(x) = f(y)$ for each function in the space implies $x = y$, and for each x in X, the linear functional $f \mapsto f(x)$ is continuous.

A functional Hilbert space whose functions are analytic on the underlying set will usually be called a Hilbert space of analytic functions.

From now, we assume that $\mathcal{H}$ is a Hilbert space of functions analytic on the open unit disc $\mathbb{D}$ such that for each $\lambda \in \mathbb{D}$ the linear functional e_λ of evaluation at λ is bounded on $\mathcal{H}$. Moreover, the constant function 1 and the identity function $f(z) = z$ are in $\mathcal{H}$.

Theorem 1.1.2. *([9], The Riesz Representation Theorem) Let $L : \mathcal{H} \to \mathbb{C}$ be a bounded linear functional. Then there is a unique vector h_0 in $\mathcal{H}$ such that $L(h) = \langle h, h_0 \rangle$ for every h in $\mathcal{H}$. Moreover, $\|L\| = \|h_0\|$.*

Let K_x be the linear functional for evaluation at x, that is, $K_x(f) = f(x)$. For functional Hilbert spaces, the Riesz Representation Theorem implies that there is a function (which we will usually call K_x) in the Hilbert space that induced this linear functional: $f(x) = \langle f, K_x \rangle$. In this case, the functions K_x are called the reproducing kernels.

Theorem 1.1.3. *([9], The Closed Graph Theorem) If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $A : \mathcal{X} \to \mathcal{Y}$ is a linear transformation such that the graph of A,*

$$graA \equiv \{x \oplus Ax : x \in \mathcal{X}\}$$

is closed, then A is continuous.

Theorem 1.1.4. *([9], Principle of Uniform Boundedness (PUB)) Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ a normed space. If $\mathcal{A} \subseteq B(\mathcal{X}, \mathcal{Y})$ such that for each x in $\mathcal{X}$, $sup\{\|Ax\| : A \in \mathcal{A}\} < \infty$, then $sup\{\|A\| : A \in \mathcal{A}\} < \infty$.*

Theorem 1.1.5. *([30], Baire Category Theorem) If X is a complete metric space, the intersection of every countable collection of dense open subsets of X is dense in X.*

1.2 A collection of spaces

In this section, we introduce some spaces of analytic functions. For a good reference on this subject see [10].

1. **Hardy space**

Definition 1.2.1. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The Hardy space $H^2(\mathbb{D})$ is the set of all analytic functions on $\mathbb{D}$ for which

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

For $f \in H^2(\mathbb{D})$, $\|f\|_2$ is the 2^{th} root of this supremum.

Definition 1.2.2. The Hardy Space $H^\infty(\mathbb{D})$ is the set of analytic functions that are bounded in $\mathbb{D}$, with supremum norm $\|f\|_\infty$.

If f is in $H^2(\mathbb{D})$, then the radial limit function of f, which is defined by

$$f^\star(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}),$$

exists and the mapping $f \mapsto f^\star$ is an isometry of $H^2(\mathbb{D})$ into a closed subspace of $L^2(\frac{d\theta}{2\pi})$. Indeed, this mapping defines an isometric isomorphism between $H^2(\mathbb{D})$ and a closed subspace of $L^2(\frac{d\theta}{2\pi})$. Using this identification, $f^\star(e^{i\theta})$ will be written as $f(e^{i\theta})$. Hence, $H^2(\mathbb{D})$ is a Hilbert space with an inner product defined by

$$\langle f, g\rangle_{H^2} = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\frac{d\theta}{2\pi}$$

and the set $\{1, z, z^2, ...\}$ is an orthonormal basis for it. Since every analytic function in the open unit disc has Maclaurin expansion that converges absolutely and uniformly on compact subsets of $\mathbb{D}$, we have

$$H^2(\mathbb{D}) = \{f = \sum_{j=0}^{\infty} a_j z^j : \sum_{j=0}^{\infty} |a_j|^2 < \infty\}.$$

2. Bergman Space

Definition 1.2.3. The Bergman space $A^2(\mathbb{D})$ is the space of all analytic functions on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi} < \infty,$$

where $dA(z) = r dr d\theta$ denotes the Lebesgue area measure on $\mathbb{D}$. Also $\|f\|_2$ is the 2^{th} root of this integral.

For f, g in $A^2(\mathbb{D})$, define:

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\frac{dA(z)}{\pi}.$$

This is an inner product on $A^2(\mathbb{D})$, and with this inner product, $A^2(\mathbb{D})$ is a Hilbert space, and $\{1, z, z^2, ...\}$ is an orthogonal set. Using the Maclaurin expansion we get:

$$A^2(\mathbb{D}) = \{f = \sum_{j=0}^{\infty} a_j z^j : \sum_{j=0}^{\infty} \frac{|a_j|^2}{j+1} < \infty\}.$$

3. Dirichlet Space

Definition 1.2.4. The Dirichlet space $\mathcal{D}$ is the set of all analytic functions on $\mathbb{D}$ for which

$$\int_{\mathbb{D}} |f'(z)|^2 \frac{dA(z)}{\pi} < \infty.$$

On this space, we have:

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \frac{dA(z)}{\pi}.$$

For f, g in $\mathcal{D}$, define:

$$\langle f, g\rangle_{\mathcal{D}} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} \frac{dA(z)}{\pi}.$$

This is an inner product on $\mathcal{D}$, and with this inner product, $\mathcal{D}$ is a Hilbert space.

We remark that the term $|f(0)|^2$ is included in the expression for the norm so that $\|1\| = 1$; some authors add $\|f\|_{H^2}^2$, instead of $|f(0)|^2$, in the definition of the norm, to accomplish $\|1\| = 1$. One can show that $\mathcal{D}$ is a functional Hilbert space, that the monomials $1, z, z^2, ...$ form an orthogonal basis for $\mathcal{D}$, and that

$$\mathcal{D} = \{f = \sum_{j=0}^{\infty} a_j z^j : \sum_{j=0}^{\infty} |a_j|^2 (j+1) < \infty\}.$$

Remark 1.2.5. There are the following relations between the above spaces:

$$\mathcal{D} \subseteq H^2 \subseteq A^2$$

and

$$H^{\infty} \subseteq H^2.$$

4. Weighted Hardy Space

Definition 1.2.6. A Hilbert space $\mathcal{H}$ whose vectors are functions analytic on the unit disc will be called a weighted Hardy space if the monomials $1, z, z^2, ...$ constitute a complete orthogonal set of non-zero vectors in $\mathcal{H}$.

The assumption that each function in $\mathcal{H}$ is analytic in $\mathbb{D}$ and not some smaller set is a pertinent part of the definition. The term "complete" is used here in its inner product space sense and the completeness is equivalent to the density of the polynomials in $\mathcal{H}$. We will usually assume that the norm satisfies the normalization $\|1\| = 1$. Writing $\beta(j) = \|z^j\|$, the orthogonality is easily seen to imply that the norm on $\mathcal{H}$ is given by

$$\|\sum_{j=0}^{\infty} a_j z^j\|^2 = \sum_{j=0}^{\infty} |a_j|^2 \beta(j)^2$$

and the inner product by

$$\langle \sum_{j=0}^{\infty} a_j z^j, \sum_{j=0}^{\infty} c_j z^j \rangle = \sum_{j=0}^{\infty} a_j \overline{c_j} \beta(j)^2.$$

The weighted Hardy space with weight sequence $\beta(j)$ will be denoted $H^2(\beta)$ or $H^2(\beta, \mathbb{D})$ if needed for clarity.

As it is mentioned in the beginning of this section, the Hardy, Bergman and Dirichlet spaces can be introduced as weighted Hardy spaces. Before discussing this point of view, we first remark that if the series $\sum_j a_j$ converges, then the root test implies that $\limsup_{j\to\infty} |a_j|^{\frac{1}{j}} \leq 1$.

Now, consider a weighted Hardy space $H^2(\beta)$ whose weight sequence is given by $\beta(j) = 1$ or $\beta(j) = (j+1)^{\frac{-1}{2}}$ or $\beta(j) = (j+1)^{\frac{1}{2}}$. Take $f(z) = \sum_{j=0}^{\infty} \hat{f}(j) z^j$

in $H^2(\beta)$. It is easily seen that $\limsup_{j\to\infty}(|\hat{f}(j)|^2\beta(j)^2)^{\frac{1}{j}} \leq 1$ and so in each of the above three cases $\limsup_{j\to\infty}(|\hat{f}(j)|)^{\frac{1}{j}} \leq 1$. Consequently, the radius of convergence of the series $f(z) = \sum_{j=0}^{\infty} \hat{f}(j)z^j$ is at least 1. This, in turn implies that f is analytic in the open unit disc $\mathbb{D}$. Thus, the classical Hardy, Bergman and Dirichlet spaces are weighted Hardy spaces with $\beta(j) = 1$, $\beta(j) = (j+1)^{\frac{-1}{2}}$ and $\beta(j) = (j+1)^{\frac{1}{2}}$, respectively.

Definition 1.2.7. The function

$$k(z) = \sum_{j=0}^{\infty} \frac{z^j}{(\beta(j))^2}$$

is called the generating function for the weighted Hardy space $H^2(\beta)$.

At first note that the generating function is analytic on the unit disc. The analyticity is a consequence of our assumption in the definition of weighted Hardy space that all the functions of $H^2(\beta)$ are analytic in $\mathbb{D}$.

Lemma 1.2.8. *If k is the generating function for a weighted Hardy space, then k is analytic on the open unit disc.*

Proof. See([10], page 16). □

Theorem 1.2.9. *Let $H^2(\beta)$be a weighted Hardy space. For each point ω in the open unit disc, evaluation of functions in $H^2(\beta)$ at ω is a bounded linear functional and, for all f in $H^2(\beta)$, $f(\omega) = \langle f, K_\omega \rangle$ where $K_\omega(z) = k(\overline{\omega}z)$. Moreover, $\|K_\omega\|^2 = k(|\omega|^2)$.*

Proof. See([10], page 17). □

Definition 1.2.10. ([10]) The automorphism of the unit disc is the one-to-one analytic map of the disc onto itself.

It is elementary to prove that if φ is a automorphism of the unit disc, then

$$\varphi(z) = \lambda\frac{a-z}{1-\overline{a}z}$$

where $|\lambda| = 1$ and $|a| < 1$.

Every disc automorphism is an automorphism of the Riemann sphere and has two fixed points on the sphere, counting multiplicity.

If $a = 0$, then $\varphi(z) = -\lambda z$ and has $z = 0$ and $z = \infty$ as its fixed points.

If $a \neq 0$, then from $\varphi(z) = z$, we have:

$$-\overline{a}z^2 + (1+\lambda)z - a\lambda = 0.$$

Thus the product of two roots of this equation is $\frac{\lambda a}{\overline{a}}$, which $|\frac{\lambda a}{\overline{a}}| = 1$. Therefore, both fixed points of φ can not lie in $\mathbb{D}$, or outside of $\overline{\mathbb{D}}$, or one in $\mathbb{D}$, and the other on $\partial\mathbb{D}$. Thus, the automorphisms are classified according to the location of their fixed points:

1. Elliptic, if one fixed point is in the disc and the other is in the complement of the closed disc, for example $\varphi(z) = iz$ which has fixed point 0 and ∞.

2. Hyperbolic, if both fixed points are on the unit circle, for example $\varphi(z) = \frac{z+0.5}{1+0.5z}$ which has fixed points ± 1.

3. Parabolic, if there is one fixed point on the unit circle (of multiplicity 2), for example, $\varphi(z) = \frac{[(1+i)z-i]}{[iz+1-i]}$ which has fixed point 1, with multiplicity 2.

Definition 1.2.11. ([10]) For ζ on the unit circle and $\alpha > 1$, we define a nontangential approach region at ζ by

$$\Gamma(\zeta, \alpha) = \{z \in \mathbb{D} : |z - \zeta| < \alpha(1 - |z|)\}.$$

Definition 1.2.12. A function f is said to have a nontangential limit at ζ if $\lim_{z \to \zeta} f(z)$ exists in each nontangential region $\Gamma(\zeta, \alpha)$.

Definition 1.2.13. We say φ has a finite angular derivative at ζ on the unit circle if there is η on the circle so that $\frac{(\varphi(z)-\eta)}{z-\zeta}$ has a finite nontangential limit as $z \to \zeta$. When it exists (as a finite complex number), this limit is denoted by $\varphi^{'}(\zeta)$.

Theorem 1.2.14. *([33], Denjoy- Wolff) Suppose that φ is an analytic mapping of the disc into itself, which is not an elliptic automorphism.*

(a) If φ has a fixed point $p \in \mathbb{D}$, then the iterates φ_n of φ converge to p uniformly on compact subsets of $\mathbb{D}$.

(b) If φ has no fixed point in $\mathbb{D}$, then there is a point $p \in \partial\mathbb{D}$ such that the iterates φ_n of φ converge to p uniformly on compact subsets of $\mathbb{D}$.

Furthermore, p is boundary fixed point of φ; and the angular derivative of φ exists at p, and $0 < \varphi^{'}(p) \leq 1$.

(c) Conversely, if φ has a boundary fixed point p in which $\varphi^{'}(p) \leq 1$, then φ has no fixed point in $\mathbb{D}$, and the iterates φ_n of φ converge to p uniformly on compact subsets of $\mathbb{D}$.

Definition 1.2.15. The limit point p of preceding Theorem will be referred to as the Denjoy-Wolff point of φ.

Remark 1.2.16. In fact Denjoy-Wolff point of φ is a fixed point of φ in $\overline{\mathbb{D}}$ with $|\varphi^{'}(p)| \leq 1$.

Lemma 1.2.17. *([10], Lemma 2.66) If φ is an analytic mapping of the disc into itself with Denjoy-Wolff point p on the circle and $\varphi^{'}(p) < 1$, then for any compact set K in $\mathbb{D}$, there is a nontangential approach region containing all the iterates $\varphi_n(K)$.*

1.3 Hypercyclic and supercyclic tuples of operators

Definition 1.3.1. An n-tuple of operators is a finite sequence of length n of commuting continuous linear operators $T_1, T_{2,}, ..., T_n$ acting on a locally convex topological vector space X.

Definition 1.3.2. ([15]) We denote the semigroup generated by a tuple $T = (T_1, ..., T_n)$ by $\mathcal{F}_T = \{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n} : k_i \geq 0, i = 1, 2, ..., n\}$ and the orbit of x under the tuple T by $orb(T, x) = \{Sx : S \in \mathcal{F}_T\}$. Also let $\mathcal{F}_T^p = \{\lambda S : \lambda \in \mathbb{C}, S \in \mathcal{F}_T\}$.

Definition 1.3.3. An n-tuple $T = (T_1, T_2, ..., T_n)$ is called hypercyclic, if there exists an element $x \in X$ such that $orb(T, x) = \{Sx : S \in \mathcal{F}_T\}$ is dense in X. In this case x is called a hypercyclic vector for T.

Definition 1.3.4. A vector $x \in X$ is called a supercyclic vector for an n-tuple

$T = (T_1, T_2, ..., T_n)$, if the set $\{Sx : S \in \mathcal{F}_T^p\}$ is dense in X, and T is said to be a supercyclic n-tuple of operators.

The above definitions generalize the hypercyclicity and supercyclicitiy of a single operator to a tuple of operators.

The cyclicity of operators have received a good deal of attention in recent years. The reference [2] provides an overview of many results that are known.

Proposition 1.3.5. *([15], Proposition 2.4) Suppose that $T = (T_1, ..., T_n)$ is a hypercyclic tuple on a separable Banach space X. Then every non-zero orbit of $T^* = ({T_1}^*, ..., {T_n}^*)$ is unbounded.*

Remember, if $\mathcal{F}$ is a set of operators on a separable Banach space X, then $\mathcal{F}$ is called hypercyclic if there exists a vector $x \in X$ such that the set $\{Tx : T \in \mathcal{F}\}$ is dense in X.

Proposition 1.3.6. *([15], Proposition 2.3) If $\mathcal{F}$ is a set of operators on a separable Banach space X, then $\mathcal{F}$ is hypercyclic if for any two open sets U, V there exists $T \in \mathcal{F}$ such that $T(U) \cap V \neq \emptyset$.*

Proposition 1.3.7. *(Hypercyclicity Criterion [15]) Suppose that (T_1, T_2) is a pair of operators on a separable Banach space Z. Suppose also that there exist two strictly increasing sequences of positive integers $\{n_j\}$ and $\{k_j\}$, dense sets X and Y in Z and functions $S_j : Y \to Z$ such that :*

(1) For each $x \in X$, $T_1^{n_j} T_2^{k_j} x \longrightarrow 0$ as $j \longrightarrow \infty$;

(2) For each $y \in Y$, $S_j y \longrightarrow 0$ as $j \longrightarrow \infty$;

(3) For each $y \in Y$, $T_1^{n_j} T_2^{k_j} S_j y \longrightarrow y$ as $j \longrightarrow \infty$.

Then (T_1, T_2) is hypercyclic pair.

Proof. If U and V are two non-empty open sets in Z, then choose $x \in X \cap U$ and $y \in V \cap Y$ and let $z_j = x + S_j y$. Then z_j and $T_1^{n_j} T_2^{k_j} z_j = T_1^{n_j} T_2^{k_j} x + T_1^{n_j} T_2^{k_j} S_j y$ converge to x and y, respectively. Thus for large j we have $z_j \in U$ and $T_1^{n_j} T_2^{k_j} z_j \in V$. Thus Proposition 1.3.6 implies that the pair (T_1, T_2) is hypercyclic. □

Theorem 1.3.8. *([20], Theorem 4.9) Let Ω denotes a domain (connected, open set) in $\mathbb{C}$, and H is a Hilbert space of analytic functions on Ω. Suppose every bounded analytic function φ on Ω is a multiplier of H, with $\|M_\varphi\| = \|\varphi\|_\infty$. Then for each such φ the operator M_φ^* is hypercyclic if and only if $\varphi(\Omega)$ intersects the unit circle.*

Lemma 1.3.9. *([2], Lemma 1.27) Let $a, b, \lambda, \mu \in \mathbb{C}$. The set $\mathbb{C}.\{(a\lambda^n, b\mu^n) : n \in \mathbb{N}\}$ is not dense in $\mathbb{C}^2$.*

1.4 Hypercyclic diagonal matrix tuples

Definition 1.4.1. An $n \times n$ matrix with $a_{ij} = 0$ for all $i \neq j$ is called a diagonal matrix.

Definition 1.4.2. An $n \times n$ matrix A is diagonalizable, if there exists an invertible matrix P such that matrix $P^{-1}AP$ is a diagonal matrix.

Theorem 1.4.3. *([15], Theorem 3.4) For each* $n \geq 1$*, there exists a hypercyclic* $(n+1)$*-tuple of diagonal matrices on* $\mathbb{C}^n$*.*

Theorem 1.4.4. *([15], Theorem 3.6) There does not exist a hypercyclic n-tuple of diagonalizable matrices on* $\mathbb{C}^n$*.*

Proposition 1.4.5. *([15], Corollary 4.2) If* $a, b > 1$ *are relatively prime integers, then* $\{\frac{a^n}{b^k} : n, k \in \mathbb{N}\}$ *is dense in* $\mathbb{R}^+$*.*

Definition 1.4.6. ([17], Definition 1) Let X and Y be topological spaces and $T_\imath : X \to Y$ $(\imath \in I)$ continuous mappings. Then an element $x \in X$ is called universal element (for the family $(T_\imath)_{\imath \in I}$) if the set

$$\{T_\imath x : \imath \in I\}$$

is dense in Y. The set of universal elements is denoted by $\mathcal{U} = \mathcal{U}(T_\imath)$. The family $(T_\imath)_{\imath \in I}$ is called universal if it has a universal element.

Theorem 1.4.7. *([17], Theorem 1 (The Universality Criterion), pp. 348-349) Suppose that* X *is a Baire space and* Y *is second-countable. Then the following assertions are equivalent:*

(i) The set $\mathcal{U}$ *of universal elements is dense in* X*.*

(ii) To every pair of non-empty open subsets U *of* X *and* V *of* Y *there exists some* $\imath \in I$ *with*

$$T_\imath(U) \cap V \neq \emptyset.$$

If one of these conditions holds, then $\mathcal{U}$ *is a dense* G_δ*-subset of* X*.*