

IN THE NAME OF GOD

**H. OPTIMAL CONTROL AND SOME OF ITS APPLICATIONS**

BY  
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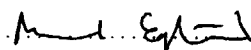
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Dedicated to  
my

MOTHER & FATHER

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## ABSTRACT

### $H_\infty$ OPTIMAL CONTROL AND SOME OF ITS APPLICATIONS

BY

MOHAMMAD GANJVAR

$H_\infty$  (pronounced as 'H-infinity') control theory has been an extensive research topic in the last ten years, and as a result, there are so many approaches to the state-space  $H_\infty$  control theory. An elegant proof was first given in Doyle, et al. 1989 [9].

The present work, which is basically an overview of the subject starts with a literature survey of  $H_\infty$  control theory that can be useful for researchers (Chapter 1). This follows by the preliminaries to understand the concepts (Chapter 2 and 3) and solving (Chapter 4) of  $H_\infty$  problem. Chapter 5 obtains the performance indices of optimal and robust control.

This relatively new approach to feedback design is explained through a solution for special case (state-space problem) given in Chapter 6. A better algorithm, however, and one which can be applied to general  $H_\infty$  problems, is given in Section 6.7 (without any derivation). Most of the contents are not new but we believe that the presentation given here may be of interest to those who are interested in learning  $H_\infty$  control with minimal mathematical background.

Finally, in Chapter 7 simple solved examples, illustrate this algorithm.

# TABLE OF CONTENTS

Content	Page
<b>ABBREVIATIONS</b>	<b>IX</b>
<b>LIST OF FIGURES</b>	<b>X</b>
<b>CHAPTER 1 HISTORY AND LITERATURE SURVEY</b>	<b>1</b>
1.1 The 70's, Briefly	1
1.2 The 80's and the Rise of $H_\infty$	3
1.3 Robust LQG at Last?	6
<b>CHAPTER 2 INTRODUCTIONS</b>	<b>8</b>
2.1 General Control Problem	8
2.1.1 Experimental Phase	8
2.1.2 Simulation Phase	10
2.1.3 Theoretical Phase	12
2.2 State-Space Model	13
2.2.1 Introduction	13
2.2.2 Transfer Function of Linear System	14
2.2.3 State Variable of Dynamic System	15
2.2.4 The State Space Model	16
<b>CHAPTER 3 MATHEMATICAL BACKGROUND</b>	<b>18</b>
3.1 Algebraic Structures	18
3.1.1 Field	18
3.1.2 Linear Vector Space	19
3.1.3 Metric, Norm, and Inner Product	20

3.2	Function Spaces	23
3.2.1	Introduction	23
3.2.2	Banach and Hilbert Spaces	23
3.2.3	Operators and Signal Spaces	24
3.2.3.1	Time Domain Definition	24
3.2.3.2	Frequency Domain Definition	25
3.3	Signal and System Space	26
3.3.1	Introduction	26
3.3.2	Time Domain Spaces	27
3.3.2.1	Hilbert Space for Signals	27
3.3.3	Frequency Domain Hilbert Spaces	30
3.3.3.1	The Fourier Transform	30
3.3.3.2	The Hardy Spaces: $H_2$ and $H_2^\perp$	33
3.3.3.3	Decomposing $L_2$ Space	34
3.3.3.4	The $H_2$ System Norm	36
3.3.4	The $H_\infty$ Norm: SISO Systems	38
3.3.4.1	Transfer Function Characterization of the $H_\infty$ Norm	39
3.3.4.2	Transfer Function Spaces	42
3.3.4.3	The Small Gain Theorem	43
3.3.5	The $H_\infty$ Norm: MIMO System	45
3.3.5.1	The $L_\infty$ and $H_\infty$ Norm for Transfer Function Matrices	46
<b>CHAPTER 4</b>	<b>BASIC CONCEPTS (PRELIMINARIES)</b>	<b>47</b>
4.1	Singular Value Decomposition (SVD)	47
4.1.1	Pseudoinverse	48
4.2	Controllability	49

4.2.1	Controllable Subspace	51
4.3	Observability	54
4.3.1	Observer Design	55
4.3.2	Transformation to Observer Form	56
4.4	Riccati Equation	58
<b>CHAPTER 5</b>	<b>OPTIMAL CONTROL</b>	64
5.1	Performance Indices	64
5.2	Optimal Control Systems	66
5.3	Quadratic Optimal Control	67
5.4	Performance	71
5.5	Uncertainty	73
5.5.1	Plant Uncertainty	73
5.5.2	Multiplicative Perturbation	74
5.6	Robustness to Additive Perturbations	75
5.7	Mixed Performance and Robustness Objective	78
5.8	Singular-Value Loop-Shaping: Mixed-Sensitivity Approach	79
<b>CHAPTER 6</b>	<b><math>H_\infty</math> CONTROL</b>	84
6.1	Introduction	84
6.2	The Standard $H_\infty$ Problem	85
6.3	Some key Facts	88
6.4	Proof of Theorem 1	89
6.5	Computing the $H_\infty$ Norm	93
6.6	$H_\infty$ Full Information and Full Control Problems	94
6.6.1	Full Information Problem	95
6.6.2	Full Control Problem	96

6.6.3	Disturbance Feedforward Problem	97
6.6.4	Output Estimation Problem	98
6.7	The Glover-Doyle Algorithm for General $H_\infty$ Problem	98
<b>CHAPTER 7</b>	<b>APPLICATION OF <math>H_\infty</math> TO SAMPLE PROBLEMS</b>	106
7.1	Examples	107
7.2	Figures (Results)	112
	<b>APPENDICES</b>	118
	<b>REFERENCES</b>	129
	<b>ABSTRACT AND TITLE PAGE IN PERSIAN</b>	



7.8	Response of Nominal Plant and Worst Case Plane (10% Increase in Masses, Springs and Dampers) of Example 2 to Square Input with control	114
7.9	Response to Pulse Generator in Example 2	115
7.10	$S$ and $1/W1$ in Example 3	115
7.11	Complementary Sensitivity and $1/W3$ in Example 3	116
7.12	Altitude Variations with respect to impulse in 1: Spoiler Angle, 2: Forward Acceleration and 3: Elevator Angle (with and without control)	116
7.13	Response of output 2: Forward Speed to impulse input in 1,2,3	117
7.14	Pitch angle in effect of impulse to inputs 1,2 and 3	117

## LIST OF FIGURES

Figure		Page
2.1	Translation of a Physical System to a FDLTI Model	13
5.1	Simple Feedback System	64
5.2	A control system in terms of $x$ and $u$	66
5.3	Unity Feedback Loop	71
5.4	Standard representation of plant under feedback Control	75
5.5	Feedback around a plant with additive uncertainty	77
5.6	Feedback loop with the same stability as that in Figure 5.5.	77
5.7	Block Diagram of the Multivariable Feedback Control System	79
5.8	Additive/Multiplicative uncertainty	81
5.9	Singular value specifications on $S$ and $T$	83
6.1	Block Diagram of General Loop Structure for $H_2$ and $H_\infty$ Problems	85
6.2	Controller for a plant which is not strictly proper	104
7.1	Mass-Spring-Damper System of Example 1	107
7.2	Tow Crane System of Example 2	108
7.3	Impulse Response of Example 1	112
7.4	Sensitivity and $1/W_1$ (Example 2)	112
7.5	$T$ and $1/W_3$ (Example 2)	113
7.6	Impulse Response of Crane 1 in Example 2	113
7.7	Step Response of Crane 1 in Example 2	114

## ABBREVIATIONS

$\gamma$	$H_\infty$ Bound or Lock Inertia Number
$\sigma_i$	i-th Singular Value
$\bar{\sigma}$	Maximum Singular Value
$\underline{\sigma}$	Minimum Singular Value
<b>A, B...</b>	Matrices (Bold Face Capital Words)
<b>x, u...</b>	Vectors (Bold Face)
$e_{ss}$	Steady-State Error
<b>G</b>	Plant
<b>H</b>	Hamiltonian Matrix
<b>K</b>	Generalized or Feedback System
$L_2$	Space of Signals with Finite 2-Norm
$H_2$	Subspace of $L_2$ Composed on Analytic Functions in $\text{Re}(s) > 0$
$H_2^\perp$	Subspace of $L_2$ Composed on Analytic Functions in $\text{Re}(s) < 0$
<b>P</b>	Generalized Plant
$S$	Closed-loop Sensitivity
$t_f$	Finite Time
$T$	Closed-loop Complementary Sensitivity
$W_i$	Weighting Matrices
$\Delta$	Plant Uncertainty
$\ \cdot\ $	Vector Euclidian Norm
$\ \cdot\ _2$	Signal or System 2-norm
$\ \cdot\ _\infty$	System $\infty$ -norm
$\langle \cdot, \cdot \rangle$	Inner Product

## CHAPTER 1

### HISTORY AND LITERATURE SURVEY

#### Abstract

This part will very briefly review the history of the relationship between modern optimal control and robust control. The latter is commonly viewed as having arisen in reaction to certain perceived inadequacies of the former. More recently, the distinction has effectively disappeared. Once controversial notions of robust control have become thoroughly main streams, and optimal control methods permeate robust control theory. This has been especially true in  $H_\infty$  theory.

The primary focus of this chapter, which will serve as a short introduction to this thesis. Much of this is taken directly from [68], which also has most of the references.

#### 1.1 The 70's, briefly

We could trace the origins of robust control almost arbitrarily far back in time, since robustness has always been the point of feedback, but we will start this narrative more recently by quickly recalling the controversy about LQG robustness in the mid to late 70's. One of the foundations of modern control theory was optimal control, which was tremendously successful in a variety of applications. The modern optimal control sample for feedback design, i.e. the LQG problem, however, had relatively little impact on practical control design. One

of the critiques of LQG was that it did not directly address many issues that were already well understood in at least some limited way in classical control, and gain and phase margins were often pointed to as an example of this. The LQG proponents could, however, point to certain guaranteed properties of LQG regulators as indications of inherent robustness. As we now know, these guarantees were of no practical value, and indeed the whole notion of guaranteed margins is unacceptable.

At about the same time, singular values or the  $H_\infty$  norm was proposed for robustness analysis of multi-variable systems. This point of view added necessity to the small gain methods of the 1960s [65,63,52].

That is, whereas small gain gave sufficient conditions for stability for a set of uncertainty, the robust control interpretation was that the same condition was necessary and sufficient for a particular set. This emphasis on necessity motivated much study of the potential conservativeness of robustness measures and techniques for reducing it.

One of the motivations for the original introduction of  $H_\infty$  methods by Zames [66] was to emphasize plant uncertainty. The  $H_\infty$  norm was found to be appropriate for specifying both the level of plant uncertainty and the signal gain from disturbance inputs to error outputs in the controlled system. The  $H_\infty$  norm gives the maximum energy gain (the induced,  $L_2$  system gain), or sinusoidal gain of the system. This is in contrast to the  $H_2$  norm, which gives the variance of the output given white noise disturbances. The robust stability consequence was the main motivation for the development of  $H_\infty$ .

methods rather than the worst-case signal gain. We compromised on performance to get one norm that let us do everything. With this compromise, we could then talk about robust performance with structured uncertainty.

## 1.2 The 80's and the rise of $H_\infty$

The synthesis of controllers that achieve an  $H_\infty$  norm specification gives a well-defined mathematical problem whose solution became a major research focus in the 1980s. Most of the original solution techniques were in an input-output setting and involved analytic functions (Nevanlinna-Pick interpolation) or operator-theoretic methods (Sarason [56], Ball-Helton[3]) and such derivations involved a fruitful collaboration between operator theorists and control engineers. Indeed,  $H_\infty$  theory seemed to many to signal the beginning of the end for the state-space methods, which had dominated control for the previous 20 years. Unfortunately, the standard frequency-domain approaches to  $H_\infty$  started running into significant obstacles in dealing with multi-input multi-output (MIMO) systems, both mathematically and computationally, much as the  $H_2$  (or LQG) theory of the 1950's had.

Not surprisingly, the first solution to a general rational MIMO  $H_\infty$  optimal control problem [8], which we will refer to as the 1984 approach, relied heavily on state-space methods, although more as a computational tool than in any essential way. The procedure involved state-space inner/outer and coprime factorization of transfer function matrices, which reduced the problem to a Nehari/Hankel norm problem solvable by the state-space method in [17]. Both [14] and

[15] gave expositions of this approach, which in a mathematical sense “solved” the general rational problem. Much of the subsequent work in  $H_\infty$  control theory focused on the  $2 \times 2$ -block problem that was a central part of this solution, either in the model-matching or general distance forms. Unfortunately, the associated complexity of computation was substantial, involving several Riccati equations of increasing dimension, and formulae for the resulting controllers tended to be very complicated with high state dimension. Encouragement came from [34] and [35] who showed, for problems transformable to  $2 \times 1$ -block problems, that a subsequent minimal realization of the controller has state dimension no greater than that of the plant. This suggested the likely existence of similarly low dimension optimal controllers in the general  $2 \times 2$  case. Additional progress came from [2, 27, 13, 24, 32, and 31].

Simple state-space  $H_\infty$  controller formulae were first announced in Glover and Doyle [18] (after some sustained manipulation). However the very simplicity of the new formulae and their similarity with the  $H_2$  ones suggested a more direct approach. Independent enhancement for a simpler approach to the  $H_\infty$  problem came from papers by Khargonekar, Petersen, Rotea, and Zhou [28, 29]. They showed that for the state-feedback  $H_\infty$  problem one could choose a constant gain as a (sub)optimal controller. In addition, a formula for the state-feedback gain matrix was given in terms of an algebraic Riccati equation. Also, these papers established connections between  $H_\infty$  optimal control, quadratic stabilization, and linear-quadratic differential games. They showed that solving an algebraic Riccati