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The universal semigroup compactifications with respect to some algebraic properties

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IN THE NAME OF GOD

THE UNIVERSAL SEMIGROUP COMPACTIFICATIONS

WITH RESPECT TO SOME ALGEBRAIC PROPERTIES

By:

HAMID REZA EBRAHIMI-VISHKI

A thesis submitted to the Faculty of Mathematics of the Ferdowsi University of Mashhad in partial satisfaction of the requirements for the degree of the Doctor of Philosophy

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PREFACE

The best mathematics is the most mixed-up mathematics, those disciplines in which analysis, algebra and topology all play a vital role.

A .D. Wallace

Harmonic analysis is primarily the study of functions and measures on the topologico-algebraic structures. Here, the underlying structure is given by a semitopological semigroup, i.e. a semigroup with a topology rendering continuous th left and right translations.

The general theme on which this thesis is based, is the notion of "*Semigroup Compactification*", that is a compact, Hausdorff, right topological semigroup which contains a dense continuous homomorphic image of a given semitopological semigroup. This notion, which in a sense, is a generalization of the classical Bohr (almost periodic) compactification of the additive group of reals, has been produced in several principal ways.

The first method is based on the oprator theroy, whose techniques have been originated in an influential paper of de Leeuw and Glicksberg in 1961, [**17**]. They used some semigroups of translation operators on the Banach spaces of almost periodic and weakly almost periodic functions (where the former was first defind by Bohr in 1925-6, [**10**]; and the latter was first introduced by Eberlein in 1949, [**18**]), to construct the almost periodic and weakly almost periodic compactifications, for a given semitopological semigroup with identity.

In the second approch to the theory, whose ideas are given by Berglund and Hofmann in their 1967 monograph, [4], the co-adjoint functor theorem of category theory is employed. Their methods extend the existence of almost periodic and weakly almost periodic compactifications for an arbitrary semitopological semigroup.

A third approach to the theory, whose seminal technique is due to Loomis, 1953 [**50**], is based on the Gelfand-Naimark theory of commutative C^*-algebras. Indeed, the spectrum of a certain C^*-algebra of functions (called m-admissible algebra) on a given semitopological semigroup, is its compactification. More precisely, the compactifications and the m-admissible function algebras are in a one-to-one correspondence, see (7.1).This point of view provides a unified framework for diverse ideas, so in this regard it is a fruitful method. It provides a convenient and efficient way to employ compactifications to obtain information about the corresponding function algebras; the functional analytic properties of an m-admissible function algebra may be examined via the algebraic or/and topological properties of the associated compactification. For example, a classical theorem of Bohr, which asserts that every almost periodic function on reals can be uniformly approximated by trigonometric polynomials, may be obtained immediately from the Peter-Weyl theorem via the device of almost periodic compactification, see (8.47 (iii)). As another example of the utility of compactifications from this point of view, the amenability of a certain function algebra may be followed by the amenability of the corresponding compactification, see (1.14), and also (8.40). Another important advantage of this point of view is that, it provides a parallel theory of affine compactification. Admittedly these three approaches have different flavours and use different techniques, yet we think that an interaction between them might be fruitful.

The main theme of this thesis is the inspection of the universal properties that a compactificatin may enjoy. For instance, it has been shown that the almost periodic and weak almost periodic compactifications are universal with respect to the properties of being a topological semigroup and a semitopological semigroup, respectively. Indeed, the main problem is to characterize some universal P-compactifications (whose existence for a variety of properties P, is given in terms of subdiect products, [**40**]) in terms of m-admissible function algebras. For example, as we have mentioned earliear, the universal semitopological and the universal topological compactifications are characterized in terms of the m-admissible algebras of weakly almost periodic and almost periodic functions, respectively; also the strongly almost periodic functions characterize the universal topological group compactification; and some types of distal functions are used to construct the universal group compactification, etc. We carry on these investigations to give such a characterizations for the universal semilattice compactification, the universal nilpotent group compactification, and some other universal compactifications whose properties satisfy a variety of semigroups or groups.

The material falls into four chapters, with an appendix synopsizing topics in tabular and diagram forms, to strees the unity of the subject. Each chapter consists of some sections and start with a very short introduction, which describes the material included. We have tried to make the text self-contained, as far as possible, however we have broken this principle in a few places which have not been essential for later developments. For this, chapter one is devoted to a moderate discussion on preliminaries, according to our requirments. Some sections of this chapter end with historical notes, our aim has not been to write a thorough account, but we hope that the comments are fairly illuminitive. Section 8 is the heart of chapter one; it presents a lot of well-known m-admissible function algebras and examines the universal property of the associated compactifications. Almost all of the material in this chapter, that belongs to the general folklore, are quoted (without proofs) from the very extensive monograph, "*Analysis on semigroups*, Berglund et al. [**7**]", which in the author's opinion, is a mini-goldmine in its own area, and all of "Analytic semigroupers" could squeeze under its umbrellah. However there are some exceptions, for example see (1.4), (3.7), (8.43), etc. Nonetheless readers who are familiar with this monograph, may start reading from chapter two, without loss of continuity.

Last three chapters are claimed to be new. Chapter two presents three m-admissible function algebras $\mathcal{AB}$, $\mathcal{BD}$, and $\mathcal{SL}$, to construct the universal abelian, band, and semilattice compactifications, respectively. The main results are (11.3), (12.3), and (12.4). Some inclusion relationships between these function algebras and the other well-known ones, presented in section 8, are made via the device of compactifications; see (12.6).

Chapter three is about the characterization of the universal nilpotent group compactification in terms of an m-admissible function algebra $\mathcal{N}_n\mathcal{G}$. The main result is (14.2).

Chapter four is intended to extend the results of chapters two and three, for a broad variety of properties. Indeed, it shall give two m-admissible function algebras $\mathcal{F}_V$ and $\mathcal{F}_\mathcal{V}$, to construct the universal V-compactification, and the universal $\mathcal{V}$-compactification, respectively, where V and $\mathcal{V}$ stand for varieties of semigroups and groups, respectively. The main results are (15.4), and (16.1). Each of last three chapters end with some problems that we belive to be open and need further research.

As prerequisites, the reader is expected to be acquainted with the general topology and elementary functional analysis; our ground rule is that, facts and concepts from Kelley [**41**], and Rudin [**66**] may be used without explanation or proof.

Finally, I take this occasion to record my debt to many people who have assisted me during the period of my research. First and foremost, my supervisor Professor M. A. Pourabdollah who deserves my profound gratitude for his enormous help, valuable guidance, and repetitious encouragement. My friend A. Sahleh with whom I have had useful conversations. Many mathematicians who have contributed to me; I mention in particular, Professors J. F. Berglund, A. Bouziad, H. D. Junghenn, A. T. M. Lau, P. Milnes, R. D. Pandian. The Iranian Ministry of Culture and Higher Education for financial support. And last (but not least) my family, specially my long-suffering parents, for their unfailing courtesy, great patience, and untiring assistance throughout my education.

Hamid Reza Ebrahimi-Vishki
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CHAPTER ONE

PRELIMINARIES

This chapter, which is falls into ten sections, is a moderate survey of the algebraic and topological theory of semigroups, emphasizing those aspects of the theory that will be used in the sequel. For notation and terminology we shall follow Berglund et al. [**7**], as far as possible. It should be emphasized that some parts of this chapter, such as (3.3),···, (3.9) and (8.48), are also claimed to be new.

1. Algebraic Semigroups

(1.1) Some types of semigroups. By a *semigroup* we shall mean a non-empty set S on which an associative binary operation, which is usually referred as the *multiplication* of S, and written multiplicatively, is defind. Hereafter S will be at least a semigroup. Every non-empty subset of S which is a semigroup under the restriction of the multiplication of S, is called a *subsemigroup* of S.

For $s, t \in S$ we write $\rho_t(s) = st = \lambda_s(t)$. For subsets A, B of S define $As = \rho_s(A)$ and $sA = \lambda_s(A)$, and $AB = \{st : s \in A, t \in B\}$.

The *center* of S which is denoted by $Z(S)$, is defined by $Z(S) = \{s \in S : st = ts \text{ forall } t \in S\}$. S is called *commutative*, or *abelian*, if $Z(S) = S$. An element $e \in S$ is said to be an *idempotent* if $e^2 = e$; the set of all idempotents of S is denoted by $E(S)$. S is said to be a *band* if $E(S) = S$. An abelian band is called a *semilattice*.

A band S is said to be a *rectangular band* if $stu = su$, for all $s, t, u \in S$. S is called *an inflation of a rectangular group* if $E(S)$ is non-void, and $seu = su$, for all $s, u \in S$ and $e \in E(S)$. Therefore a band is a rectangular band if it is an inflation of a rectangular group. An inflation of a rectangular group S which is simple (i.e. $SsS = S$ for all $s \in S$) and contains an idempotent e such that $Ss = Se$ for all $s \in Se$ (such an e is called a *minimal idempotent*), is called a *rectangular group*.

An element $1 \in S$ is said to be a *left* (resp. *right*) *identity* of S if $1s = s$ (resp. $s1 = s$), for all $s \in S$. A left identity that is also a right identity is called an *identity*.

An element $z \in S$ is called a *left* (resp. *right*) *zero* of S if $zs = z$ (resp. $sz = z$), for all $s \in S$. If all elements of S are left (resp. right) zeros, then S is called a *left* (resp. *right*) *zero semigroup*.

(1.2) Example. The set $S = \{a, b, c, d\}$ under the multiplication table,

	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	c	c	c
d	c	c	c	c

is a semigroup with a left zero that is not a right zero, and a left identity that is not a right identity. Furtheremore, $Z(S) = \emptyset$, and $E(S) = \{a, b, c\}$; see [**7**; 1.1.7].

S is said to be a *left (resp. right) group* if for each pair $s, t \in S$ there exists a unique $x \in S$ such that $xs = t$ (resp. $sx = t$). S is called *left (* resp. *right) simple* if $Ss = S$ (resp. $sS = S$), for all $s \in S$. It is easy to check that, a semigroup is a left group if and only if it is left simple and contains an idempotent, [**7**; 1.2.19].

S is called a *group* if it contains the identity 1, and for every $s \in S$ there exists $s^{-1} \in S$, such that $ss^{-1} = s^{-1}s = 1$. The next assertion reveals a connection between being group and left-right simplicity.

(1.3) Proposition. *A semigroup is a group if and only if it is both left simple and right simple*; [**7**; 1.1.17].

(1.4) Nilpotent groups. A subsemigroup of a group which is also a group, is said to be a *subgroup* of it. A subgroup T of a group S is called a *normal* subgroup if for all $s \in S$, $s^{-1}Ts = T$. Then we have the factor group S/T, which is the set of all cosets of T in S, equipped with the group multiplication $(sT)(s'T) = (ss'T)$. A group S is called *nilpotent* if there exist a finite sequence of the normal subgroups $T_0, T_1, \cdots, T_n$ including $\{1\}$ and S such that each member is a subset of its successor; i.e. $\{1\} = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_n = S$, and the factor T_{i+1}/T_i is contained in $Z(S/T_i)$ for all i. If all the T_i are distinct, the integer n is called the length of the series. The length of a shortest such series for a nilpotent group is called the *nilpotency class* of it. Obviously, a nilpotent group of class 0 is trivial, and a group is nilpotent of class 1 if and only it is abelian. We shall need the following characterization of nilpotet groups.

(1.5) Proposition. *A group S is a nilpotent group of class n if and only if for any $n+1$ elements $s_1, s_2, \cdots, s_{n+1}$ of S,* $[s_1, s_2, \cdots, s_{n+1}] = 1$; [**65**; 5.1.9].

Note that $[., ., \cdots, .]$ is the commutator of weight $n+1$ in S, which is defined recursively by the rules $[s_1, s_2] = {s_1}^{-1}{s_2}^{-1}s_1s_2$ and $[s_1, s_2, \cdots, s_n, s_{n+1}] = [[s_1, s_2, \cdots, s_n], s_{n+1}]$.

(1.6) Example. The set of all symmetries of a regular polygon with n sides ($n \geq 3$) forms a group of order $2n$ which is called the *dihedral group of order* $2n$, and is denoted by D_{2n}. In terms of presentation of groups, [**65**; 2.2], D_{2n} has the following presentation, $D_{2n} = \prec a, b \mid a^n = b^2 = (ab)^2 = 1 \succ$. Therefore one may show that $D_{2n} = \{1, a, a^2, \cdots a^{n-1}, b, ab, a^2b, \cdots, a^{n-1}b\}$.

It is simple to verify that, D_6 is not nilpotent, and D_8 is nilpotent of class 2.

(1.7) Semidirect products. A mapping θ from S into another semigroup T is called a *homomorphism* if $\theta(ss') = \theta(s)\theta(s')$ for all $s, s' \in S$. A homomorphism from S into the semigroup of all complex numbers with modulus 1 is called a *character* of S.

Let σ be a homomorphism of the semigroup T into the semigroup of all homomorphisms from S into itself; we write σ_t instead of $\sigma(t)$ for $t \in T$. Then, with multiplication $(s,t)(s',t') = (s\sigma_t(s'), tt')$, the direct product $S \times T$ is a semigroup, called a *semidirect product* of S and T, and is denoted by $S \circledcirc T$. Note that $S \circledcirc T$ reduces to the direct product semigroup if σ_t is the identity on S, for all $t \in T$. If S and T have identities (each denoted by 1), then $(1,1)$ is an identity for $S \circledcirc T$ if and only if σ_1 is the identity on S and $\sigma_t(1) = 1$, for all $t \in T$. As in the direct product case we shall denote by $p_1 : S \circledcirc T \longrightarrow S$ and $p_2 : S \circledcirc T \longrightarrow T$ the projection mappings, and by $q_1 : S \longrightarrow S \circledcirc T$ and $q_2 : T \longrightarrow S \circledcirc T$ the injection mappings, $(q_1(s) = (s,1), q_2(t) = (1,t)$). Note that, q_1, q_2 and p_2 are homomorphisms, but in general p_1 is not; and it is this latter fact which distinguishes the semidirect product from the direct product.

Using the notion of direct product, we have the following characterizations for rectangular bands, rectangular groups, and left (or right) groups.

(1.8) Proposition. *Let S be a semigroup; then* *(i) S is a rectangular band if and only if it is isomorphic to the direct product of a left zero semigroup and a right zero semigroup;* [**7**; 1.1.48],

(ii) S is a rectangular group if and only if it is isomorphic to the direct product of a rectangular band and a group; [**7**; 1.2.23]*; and*

(iii) S is a left (resp. right) group if and only if it is isomorphic to the direct product of a left (resp. right) zero semigroup and a group; [**7**; 1.2.19].

(1.9) Variety of semigroups. Let A be a non-empty set, and let F_A be the *free semigroup* on A, which is the set of all non-empty finite words in the "alphabet" A, whose multiplication is given by juxtaposition. A is called the generating set of F_A; and we usually identify each element $a \in A$ with the one-letter word a in F_A. The following proposition includes the crucial property of free semigroups.

(1.10) Proposition. *Let A be a non-empty set. Then for each semigroup S, every mapping $\theta : A \longrightarrow S$ extends to a homomorphism $\Phi : F_A \longrightarrow S$;* [**32**; 1.6.1].

Let A be countable, and let $p, q \in F_A$. Then we shall say that the semigroup S satisfies the identity $p = q$ if $\Phi(p) = \Phi(q)$ for every homomorphism $\Phi : F_A \longrightarrow S$. Informally, if we think of the elements of A as variables, we are requiring that the two words p and q give rise to equal elements of S for every possible assignment of values from S to the variables.

A class V of semigroups in which a collection E of identities is satisfied is called a *variety of semigroups.* For example, the class of semilattices is a variety, determined by $E = \{a^2 = a, ab = ba\}$. It is important to realize that two different sets of identities may determine the same variety; for instance, the variety of rectangular bands can be characterized either as $E = \{a^2 = a, abc = ac\}$ or as $E = \{aba = a\}$; [**7**; 1.1.48]. A variety of semigroups is trivially closed under the formation of direct products, subsemigroups, and homomorphic images. Moreover, we quote a result due to Birkhoff, [**9**], which asserts that these " closure properties " characterize varieties.

(1.11) **Proposition.** *A non-empty class of semigroups forms a variety if and only if it is closed under the formation of direct products, subsemigroups, and homomorphic images.*

Hence, the class of all groups does not form a variety of semigroups. Parallel to the variety of semigroups in the semigroup theory, there is the concept of "variety of groups" in the group theory, which is similarly determined in terms of free groups instead of free semigroups. More precisely, let W be a non-empty subset of the free group on a countably infinite set, (for the construction of free groups see [**65**; 2.1]); and let S be a group. The subgroup of S generated by all *values* in S of the words in W is denoted by W_S, (and is called the *verbal subgroup* of S determined by W). The class $\mathcal{V}$ of all groups S for which $W_S = \{1\}$ is called a *variety of groups* determined by the set of laws W. For example, if $W = \{[a_1, a_2]\}$, then $\mathcal{V}$ is the class of abelian groups, and if $W = \{[a_1, \cdots, a_n, a_{n+1}]\}$ then the variety $\mathcal{V}$ is the family of