In the name of God

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OBSERVATIONAL DYNAMICAL SYSTEMS

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To my family and my parent.

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Abstract

In this thesis at first, an observational consideration of fuzzy metric spaces is presented. Relative metric spaces and the topologies created by them are introduced. Then based on the spaces which they have constructed in chapter one topological entropy, minimality and transitivity in various methods are studied.
The notion of fuzzy attractor sets as the basic objects in relative semi-dynamical systems is presented. We discuss discussion on fuzzy attractor sets in the standard fuzzy metric spaces. The relation of fixed point theorem for fuzzy contracting map with its fuzzy attractor is studied. The persistence of fuzzy attractor under conjugate relation is proved. New examples in fuzzy-relative spaces are investigated.
In chapter 4 by introducing *Prevalence Whitney Embedding Theorem* another type of observation is considered. The extension of this notion for noncompact spaces is deduced.

Introduction

0.1 Introduction

A dynamical systems is any process that moves or changes in time. Dynamical systems occur in many branches of science. For example: the motion of planets, the weather, and the chemical reactions.

Mathematically dynamical systems is a pair $\{X, \varphi^t\}$, where $t \in T$ and T is an ordered set. X is a state space and $\varphi^t : X \to X$ is a family of evolution operators satisfying the properties $\varphi^0 = id$, and $\varphi^{t+s} = \varphi^t o \varphi^s$. We will consider two types of dynamical systems: those with continuous time $T = R$, and those with discrete (or integer) time $T = Z$. Systems of first type called continuous-time dynamical systems, while those of the second are discrete-time dynamical systems.

Fuzzy dynamical systems as an example of fuzzy mathematical modelling created a new way for looking into the mathematical phenomena such as topological spaces, and dynamical systems. A generalization of a fuzzy dynamical system, which is called relative semi-dynamical system has been presented to concern the dynamics on the systems related to observers, and to compare those dynamics through the mathematical methods such as entropy.

In the direction of mathematical modelling of observers, fuzzy sets [32] have play essential roles. In fact a one dimensional observer of a set X is a fuzzy set $\mu : X \to [0,1]$. Any mathematical model according to the viewpoint of an observer μ is called a relative model [17, 19, 21].

The notion of observer as a fuzzy set play an important role in biology and physics [20]. For example if we divide the population into n age groups, and we denote the number of individuals in the k^{th} group at time t by $x_k(t)$, where $k \in \{1, 2, ..., n\}$, then we have the following model for the age group population dynamics [18].

$$\begin{cases} \dot{x}_1(t) = \frac{1}{2}\sum_{k=1}^{n} \alpha_k(t) x_k(t) \\ \dot{x}_{k+1}(t) = \frac{1}{2}[\dot{s_k}(t) - \dot{e_k}(t) + \dot{c_k}(t)]x_k(t), \ \ for\ k \in \{1, 2, ..., n-1\} \end{cases}$$

where

$$\alpha_k(t) = \dot{b_k}(t) - \dot{s_k}(t)e_k(t)b_k(t) + s_k(t)\dot{e_k}(t)b_k(t) + s_k(t)e_k(t)\dot{b_k}(t)$$

$$+\dot{c_k}(t)f_k(t) + c_k(t)\dot{f_k}(t).$$

In this model $b_k(t)$, $e_k(t)$, $c_k(t)$, $f_k(t)$ and $s_k(t)$ are special observers on the set of real numbers.

There is a generalization of metric spaces by using of the fuzzy theory which is called fuzzy metric spaces [1, 26]. Our approach for generalization in this thesis is based on adding observer to this mathematical notion. There is a one to one correspondence between $[0,1]^X$, fuzzy sets, and observers. $\mu-$Fuzzy topology is a description of topology by the eyes of an observer μ.

In this thesis we study the notion of observer in dynamical systems. This study is based on two main parts. At first observers are considered, then topological spaces related to these observers are constructed. Main properties of these spaces are considered. The attractors, entropy, limit sets are presented. This process has presented in chapters 1 to 3.

In chapter four observations are investigated in the other way. The observer see all of the systems and conclude some results from it.

In chapter 1 metric spaces are generalized by using of the fuzzy theory which is called fuzzy metric spaces [1, 26]. The standard fuzzy metric will be considered, that will be used to deduce main results of attractors which are mentioned in chapter 3. In the rest we present a new axiom for fuzzy metric spaces. In a relative metric space $(X, M, *, \mu)$ the relative metric M is an observable object according to the viewpoint of the observer μ. Relative topologies created by a relative metric space and main properties of it are considered. In section 1.3, $\mu-$fuzzy metric space are presented. They are another type of observational modelling [5, 1, 29].

A few topological definitions are necessary at the beginning of our discussion in chapter 2 and chapter 3, which we present them here.

For a map $f : X \longrightarrow X$ and a point $x \in X$, the set

$$\{x,\ f(x),\ f(f(x)), ...f^n(x), ...\}$$

(when f is not invertible) or the sequence

$$\{..., f^{-1}(x), x, f(x), ...\}$$

is called the orbit of x for f. A fixed point is a point $x \in X$ such that $f(x) = x$. The set of all fixed points is denoted by $Fix(f)$. A periodic point is a point x such that $f^n(x) = x$ for some $n \in N$, that is a point in $Fix(f^n)$. Such an n is said to be a period of x. The smallest such n is called the prime period of x.

The $\alpha-limit$ set of x for ϕ_t is the set of accumulation points of $\phi_t(x)$ as $t \to -\infty$. The $\omega-limit$ set of x for ϕ_t is the set of accumulation points of $\phi_t(x)$ as $t \to \infty$. The α and $\omega-$limit set of x are its asymptotic limit sets. Another y is an accumulation point of $\phi_t(x)$ if there is a sequence $\{t_n\}$ such that $lim_{n\to\infty} t_n = \infty$ and $lim_{n\to\infty}\phi_{t_n}(x) = y$ [4].

Similarly for a map $f : X \longrightarrow X$ we can define the $\omega-$limit set of point x for f. Indeed y is an accumulation point of $f^n(x)$ if there is a sequence $\{n_i\}$ such that $lim_{i\to\infty} n_i = \infty$ and $lim_{n_i\to\infty} f^{n_i}(x) = y$. For example suppose ϕ_t is the flow of the system

$$\begin{cases} \dot{r} = r(1-r^2,) \\ \dot{\theta} = 1, \end{cases}$$

in polar coordinate. If $x \neq 0$ is inside the unite circle, then the $\alpha-$limit set of x is 0 and the $\omega-$limit set of x is the unite circle [11]. Next we shall discuss some general properties of $\omega-$limit sets. Let X be a nonempty compact set and ϕ_t be the flow of the system (or $f : X \longrightarrow X$). Let $p \in X$. Then the following properties are hold [4]:

1. $\omega(p) \neq \emptyset$,
2. $\omega(p)$ is closed,
3. $\omega(p)$ is a union of orbits of X, and
4. $\omega(p)$ is connected.

Clearly the properties above are also true for an $\alpha-$limit set.

An invariant set S for a flow ϕ_t or map f on R^n is a subset $S \subset R^n$ such that $\phi_t(x) \in S$ (or $f(x) \in S$) for all $x \in S$ and for all $t \in R$.

A closed invariant set $A \subset R^n$ is called an attracting set if there is some neighborhood U of A such that $\phi_t(x) \in U$ for $t \geq 0$ and $lim_{n\to\infty}\phi_t(x) = A$, for all $x \in U$.

The set $\bigcup_{t\leq 0} \phi_t(U)$ is the domain of attraction of A. An attracting set ultimately captures all orbits starting in its domain of attraction. A repelling is defined analogously, by replacing t with $-t$.

Domains of attraction of disjoint attracting sets are necessarily nonintersecting and separated.

In many problems we are able to find a closed connected set $D \subset R^n$ such that

$\phi_t(D) \subset D$ for all $t > 0$. In this case we can define the associated attracting set as

$$A = \bigcap_{t \geq 0} \phi_t(D).$$

For maps, a closed set A is an attracting set if it has some neighborhood U such that $lim_{n \to \infty} f^n(U) = A$. As in the case of flows we can define a closed set D such that $f(D) \subset D$ and

$$A = \bigcap_{n \geq 0} F^n(D).$$

For example [11] let $X = R^2$. Consider the system

$$\begin{cases} \dot{x_1} = -x_1 - x_2 \\ \dot{x_2} = x_1 - x_2. \end{cases}$$

We can write this system in polar coordinate as

$$\begin{cases} \dot{\rho} = -\rho \\ \dot{\theta} = 1. \end{cases}$$

So the flow of this systems is defined by $\phi_t(\rho_0,\ \theta_0) = (\rho_0 e^{-t},\ \theta_0 + t)$. So the origin $(0, 0)$ is an attractor set for flow of this continuous dynamical systems.

In chapter 2 entropy, minimality, and transitivity are discussed from an observer view-point.

In section 2.1 $\mu-$fuzzy topological entropy is defined, and we investigate some properties of it. Then we show that the relative topological entropy is an invariant object under μ-conjugate relations. At this direction relative topological entropy is presented.

We recall that a homeomorphism $f : X \longrightarrow X$ is said to be minimal if the orbit of every point $x \in X$ is dense in X or, equivalently, if f has no proper closed invariant sets. A closed invariant set is said to be minimal if it contains no proper closed invariant subsets or, equivalently, if it is the orbit closure of any of its points. In section 2.2

$\mu-$fuzzy minimality based on $\mu-$fuzzy topological spaces is considered. Developing the $\mu-$fuzzy semi-dynamical system over minimizing a polynomial on an arbitrary semi-algebraic set as a computational example is investigated. Then based on relative topological spaces, relative minimality is introduced. We finish this chapter by considering $\mu-$fuzzy transitivity.

In section 3.1 we describe the notion of attractor for a fuzzy semi-dynamical system. A comparison between attractor set and fuzzy attractor set is considered. Then fuzzy contractive mapping and fixed points and their relation with fuzzy attractor sets are presented. In section 3.2 fuzzy-relative metric spaces and it created topology is introduced. We also describe the notion of attractor set for a fuzzy semi-dynamical system when we have an observer map μ. We will investigate the topological properties of two $\mu-$semi-dynamical systems where they are different only at their observer maps. In the last section of this chapter fuzzy-relative metric space that is based on fuzzy metric space together observer $\mu : X \to [0, 1]$ is introduced.

In chapter 4 observational dynamical systems are discussed in the other way. In section 4.1 the observable dynamical systems and the notion of box-dimension are presented. Then the main theorem of this chapter(Prevalence Whitney Embedding Theorem) [24], is introduced. This theorem gives us an idea to investigate the effect of view-point to study a system.

In section 4.2 by using of the notion of base [**?**], we obtain this theorem for discrete dynamical systems $f : X \to X$, when X is a noncompact set.

Chapter 1

Fuzzy Topologies Created by Fuzzy Metric Spaces

In this chapter relative metric spaces as another generalization of metric spaces by using of an observer are introduced. A method for constructing relative topologies via a relative metric space is presented. In order to develop a mathematical model underlying uncertainty and fuzziness in a dynamical system, which is called fuzzy mathematical modelling, we are going to apply the above notions. In this case, any variation and/or approximation (physically, geometrically or topologically) on a system should be identified by an observer. Moreover, we need a method to compare between perspective of observers, also to measure the complexity and/or the uncertainty of system through viewpoint of observers. So first, we should mathematically characterize the observer. In our approach, there is a one to one correspondence between $[0,1]^X$, all functions $\mu : X \longrightarrow [0,1]$ and observers, where X denoted the base space of system. We should indicate any structure or dynamic on X as well as μ-qualify or μ-relative, which may be the multiple factors. For example: μ-fuzzy topology is the description of topological notion on X by eyes of observer μ.

1.1 Fuzzy Metric Spaces

1.1.1 A New Approach to Fuzzy Metric

Many authors have introduced the concept of fuzzy metric spaces in different ways [33, 7]. In this section, we modify the concept of fuzzy metric space introduced by Kramosil and Michalek [10] and define a Hausdorff topology on this fuzzy metric space.

a binary operation $* : (0,1] \times (0,1] \longrightarrow (0,1]$ is called a continuous $t-$norm if $*$ satisfies the following conditions;

1. $*$ is associative and commutative;

2. $*$ is continuous;

3. $a * 1 = a$ for all $a \in (0,1]$;

4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;

5. If $a * b = a * c$ then $b = c$.

The properties 4 and 5 of a continuous $t-$norm imply that if $a * b \leq a * c$ then $b \leq c$.

Example 1.1.1. The map $* : (0,1] \times (0,1] \longrightarrow (0,1]$ defined by $a * b = ab$ is a continuous $t-$norm.

Definition 1.1.2. A fuzzy metric space [2] is a triple $(X, M, *)$ where X is a nonempty set, $*$ is a continuous t-norm and $M : X \times X \times [0, \infty) \longrightarrow [0,1]$ is a mapping which has the following properties:

For every $x, y, z \in X$ and $t, s > 0$:

1) $M(x, y, t) > 0$;

2) $M(x, y, t) = 1$ if and only if $x = y$;

3) $M(x, y, t) = M(y, x, t)$;

4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;

5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is a continuous map.

Remark 1.1.1. $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t.

Example 1.1.3. [15]. In figure 1 we have illustrated a region in the city with the places A, B, C, D, E, F and the streets between them. If $x, y \in X = \{A, B, C, D, E, F\}$ and $x \neq y$ then we define $M(x, y, t) = \frac{1}{s+e^{-t}}$ where s is the number of direct streets between x and y and t is the distance between O and P. One can consider P as a parachutist. According to the parachutist's viewpoint the more brightness of the places will appear when t increase. If we define the $t-$norm as the product of real numbers, then $(X, M, *)$ is a fuzzy metric space.□

Recall [1] that in a fuzzy metric space $(M, X, *)$, we say that the sequence (x_n) in X is converges to x if $lim_{n \longrightarrow \infty} M(x_n, x, t) = 1$ for every $t > 0$. Similarly a sequence (x_n) is a cauchy sequence if $lim_{n,m \longrightarrow \infty} M(x_n, x_m, t) = 1$ for every $t > 0$ [22]. We identify $x = y$ with $M(x, y, t) = 1$, for $t > 0$.

Lemma 1.1.1.1. *[3] $M(x, y, .)$ is nondecreasing for all x, y in X.*

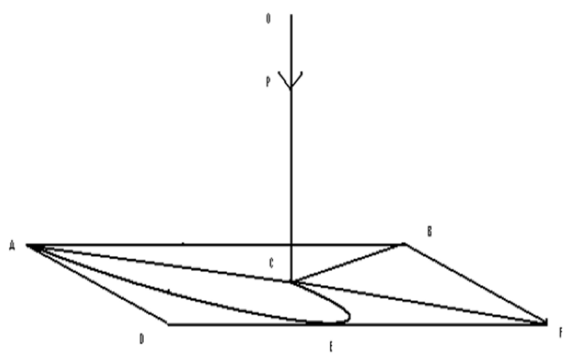

Figure 1.1: The fuzzy metric space presented in Example 1.1.3

Proof. Let $t > s$, and x, y, be two arbitrary points in X by property 4 of definition 2.1. $M(x, y, t) \geq M(x, x, t-s) * M(x, y, s)$. Then property 2 of definition 2.1. implies that $M(x, y, t) \geq 1 * M(x, y, s)$. So $M(x, y, t) \geq M(x, y, s)$. $\square$

Lemma 1.1.1.2. *In a fuzzy metric space* $(X, M, *)$:

a) whenever $M(x, y, t) > 1 - r$ *for* x, y *in* X, $t > 0$, $0 < r < 1$, *we can find a* t_0, $0 < t_0 < t$ *such that* $M(x, y, t_0) > 1 - r$.

b) For any $r_1 > r_2$, *we can find a* r_3 *such that* $r_1 * r_3 \geq r_2$ *and for any* r_4 *we can find a* r_5 *such that* $r_5 * r_5 \geq r_4$, $(r_1, r_2, r_3, r_4, r_5, \in (0, 1))$.

Proof. a) Above lemma implies that if $t_1 < t$ then $M(x, y, t_0) \leq M(x, y, t)$. If $M(x, y, t_1) = M(x, y, t)$ then $M(x, y, t_1) > 1 - r$ and proof is complete.
If $M(x, y, t_1) < M(x, y, t)$, then choose a number m such that
$max\{1-r, M(x, y, t_1)\} < m < M(x, y, t)$. Since $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is a continuous and nondecreasing map, then there exists $t_0 < t$ such that $m = M(x, y, t_0)$. So $M(x, y, t_0) > 1 - r$.
b) If $r_2 \leq k \leq r_1$ then $r_1 * r_1 \leq r_2 \leq k \leq r_1 * 1$. So by continuity of the $t-$norm $*$ there exists $r_2 \leq r_3 \leq r_1$ such that $k = r_1 * r_3$. $\square$

Example 1.1.4. Let $X = R$. Define $a * b = ab$ and $M(x, y, t) = [exp(\frac{|x-y|}{t})]^{-1}$ for all $x, y \in X$ and $t \in (0, \infty)$. Then $(M, X, *)$ is a fuzzy metric space.

(1) Clearly $M(x, y, t) = 1$ if and only if $x = y$.

(2) $M(x, y, t) = M(y, x, t)$.

(3) To prove $M(x, y, t)M(y, z, s) \leq M(x, z, t + s)$. First we prove that $| x - y | + | y - z | \leq (\frac{t+s}{t}) | x - y | + (\frac{t+s}{s}) | y - z |$. This is clear that $| x - y | ts + | y - z | ts \leq |$