To my family

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ABSTRACT

ON NUMERICAL RANGE OF SOME BOUNDED OPERATORS

BY

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In this thesis we study the numerical range of some bounded operators. For a bounded linear operator A on a Hilbert space $\mathcal{H}$, the numerical range $W(A)$ is the image of the unit sphere of $\mathcal{H}$ under the quadratic form $x \mapsto < Ax, x >$ associated with the operator, more precisely,

$$W(A) = \{< Ax, x >: x \in \mathcal{H}, \|x\| = 1\}.$$

In chapter 1 we are concerned with one of the basic problems in the theory of the numerical range which is called the numerical range attainment problem. In the end of [41] authors asked, is every compact convex set the numerical range of some operator on separable Hilbert space? In this chapter we answer this question, in some cases.

The following conjecture posed in 2000 by P. S. Bourdon and Shapiro:

The numerical range of a finite order elliptic automorphism composition operator on Hardy space is not a disc. In chapter 2 we show that this is true for

a large class of such composition operators and present some useful properties of such operators.

If A is a continuous linear operator on the normed space X, then the spatial numerical range $V(A)$ of A is defined by

$$V(A) = \{< Ax, x^* >: x \in X, x^* \in X^*, \|x\| = \|x^*\| =< x, x^* >= 1\}$$

where X^* denotes the dual space of X.

In chapter 3, we consider the spatial numerical range of operators on weighted Hardy spaces and give conditions for closedness of numerical range of compact operators. Although, the spatial numerical range need not be convex in general, we prove that it is star shaped for each finite rank operators on weighted Hardy spaces.

Finally, in chapter 4, we determine the C*-algebra numerical range of nilpotent elements.

Contents

INTRODUCTION

The thesis is organized in 4 chapters as follows:

Chapter 1 contains some notations and preliminaries which is needed during this thesis.

In chapter 2, the numerical range of composition operators on Hardy space is presented. In this chapter we consider the following conjecture, posed in 2000 by P. S. Bourdon and Shapiro: The numerical range of a finite order elliptic automorphism is not a disc, and show that this is true for a large class of such composition operators. We also present some useful properties of elliptic automorphism composition operators.

In chapter 3 we consider the spatial numerical of spacial operators on weighted Hardy space and prove that it need not be convex, in general, but it is star shaped for each finite rank operators on weighted Hardy spaces. Finally, in chapter 4, we focus on the numerical range of elements of C*-algebras.

Chapter 1

PRELIMINARIES

1 PRELIMINARIES

In this chapter we state the basic definitions and notations which are used in other chapters.

1.1 Numerical range

This is an introduction to the notion of numerical range for bounded linear operators on Hilbert space. For a bounded linear operator A on a Hilbert space $\mathcal{H}$, the numerical range $W(A)$ is the image of the unit sphere of $\mathcal{H}$ under the quadratic form $x \mapsto < Ax, x >$ associated with the operator A. More precisely,

$$W(A) = \{< Ax, x >: x \in \mathcal{H}, \|x\| = 1\},$$

and the numerical radius of A is defined by

$$r(A) = \sup\{|z| : \quad z \in W(A)\}.$$

Very little about the numerical range is obvious, here is a more-or-less complete list of what is (see [20] and [23]):

- $W(A)$ is invariant under unitary similarity, i.e., $W(A) = W(U^*AU)$ if U is a unitary operator;
- $W(A)$ lies in the closed disc of radius $\|A\|$ centered at the origin and so $r(A) \leq \|A\|$;
- $W(A)$ contains all the eigenvalues of A;

- $W(A^*) = \{\overline{z} : z \in W(A)\}$;

- If α and β are complex numbers, and A is a bounded operator on $\mathcal{H}$, then $W(\alpha A + \beta I) = \alpha W(A) + \beta$. More generally, this is even the case when taking the affine transformation: if $f(x+iy) = (a_1x+b_1y+c_1)+i(a_2x+b_2y+c_2)$ is an affine transformation of the complex plane $\mathbb{C}$, where x, y a_j, b_j and c_j, $j = 1, 2$ are all real and the latter satisfy $a_1b_2 \neq a_2b_1$, and if we define $f(A)$ to be $(a_1ReA+b_1ImA+c_1I)+i(a_2ReA+b_2ImA+c_2I)$, where $ReA = \frac{A+A^*}{2}$ and $ImA = \frac{A-A^*}{2i}$ are the real and imaginary parts of A, respectively, then

$$W(f(A)) = f(W(A)) = \{f(z) : \ z \in W(A)\}(see[17]);$$

- $W(A)$ is convex (very mysterious!);

- If $\mathcal{H}$ is finite dimensional and A is normal then $W(A)$ is the convex hull of the eigenvalues;

- $\sigma(A) \subseteq \overline{W(A)}$= the closure of $W(A)$, where $\sigma(A)$ is the spectrum of A;

- For normal operator A, $\overline{W(A)} = co(\sigma(A))$;

- If $\mathcal{H}$ is finite dimensional then $W(A)$ is compact. This fact follows from the compactness of the unit sphere of $\mathcal{H}$ and the continuity of the quadratic form associated with A;

- If $A = A_1 \oplus A_2 \oplus ... \oplus A_n$ then

$$W(A) = co \cup_{i=1}^{n} W(A_i).$$

One more important property of the numerical range map, $A \rightarrow \overline{W(A)}$, that we need in this thesis, is the continuity of it. For the convergence of compact subsets of the plane, we use the topology induced by the Hausdorff metric [23].

Let K_1 and K_2 be two compact subsets of plane. The Hausdorff distance $\Delta(K_1, K_2)$ is the minimal number r such that the closed r-neighborhood of any x in K_1 contains at least one point y of K_2 and vice versa. In other words,

$$\Delta(K_1, K_2) = \max\{\sup_{x \in K_1} \inf_{y \in K_2} |x - y|, \sup_{y \in K_2} \inf_{x \in K_1} |x - y|\}.$$

The next theorem says that the closure of the numerical range induces a continuous map on operators when the latter is endowed with the norm topology.

Theorem 1.1.1 *If $\mathcal{H}$ is a Hilbert space and $\{A_n\}$ is a sequence of bounded linear operators on $\mathcal{H}$ which converges to the bounded linear operator A in norm, Then $\overline{W(A_n)}$ converges to $\overline{W(A)}$ in the Hausdorff metric.*

1.2 Elementary examples

Example 1.2.1 A finite backward shift. Let A be the operator on $\mathbb{C}^2$ whose matrix with respect to the standard basis is

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then $W(A)$ is the closed disc of radius $\frac{1}{2}$, centered at the origin.

The next example is the generalization of before example

Example 1.2.2 Let A be the operator on $\mathbb{C}^2$ whose matrix with respect to the standard basis is

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$$

Then $W(A)$ is the closed disc of radius $\frac{|a|}{2}$, centered at the origin.

Example 1.2.3 If A is the operator on $\mathbb{C}^3$ whose matrix with respect to the standard basis is

$$\begin{bmatrix} 0 & 2r & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{bmatrix}$$

for $r > 0$ and $z \in \mathbb{C}$, then $W(A)$ is the convex hull of $\{z\}$ and the disc centered at the origin with radius r.

Example 1.2.4 Numerical range of a two by two matrix. Suppose T is a two by two matrix with distinct eigenvalues λ_1 and λ_2, to which correspond unit eigenvectors f_1 and f_2. Let $\gamma =< f_1, f_2 >$. Then:

(a) $W(T)$ is a (possibly degenerate) elliptical disc with foci at λ_1 and λ_2.

(b) The eccentricity of $W(T)$ is $\frac{1}{\gamma}$.

(c) The major axis of $W(T)$ has length $\frac{2|\lambda_1 - \lambda_2|}{\sqrt{1-\gamma^2}}$. [46]

Example 1.2.5 Let B be the backward shift operator on ℓ^2 defined by

$$B(x_0, x_1, x_2, ...) = (x_1, x_2, ...) \quad ((x_0, x_1, x_2, ...) \in \ell^2).$$

Then $W(B) = \mathbb{U}$ (the open unit disk) [46]. This example shows that when $\mathcal{H}$ is infinite dimensional it supports bounded operators with non-closed numerical range. The spectrum of an operator lies in the closure of its numerical range, and since the numerical range is convex thus the convex hull of the spectrum of an operator lies in the closure of the numerical range. By the similarity-invariance of the spectrum, its convex hull must, in fact, lie in the intersection of the closures of the numerical ranges of all the operators similar to A [46]. This leads up to the beautiful theorem of Stephan Hildebrandt which asserts that this intersection is precisely the convex hull of the spectrum.

Example 1.2.6 Sectorial Operator. Let $\theta \in (0, \frac{\pi}{2})$ and let S_θ denote the closed sector

$$S_\theta := \{0\} \cup \{z \in \mathbb{C} : |Argz| \leq \theta\},$$

in the right complex half-plane. A bounded linear operator A in $\mathcal{H}$ is called *sectorial* with vertex γ and half-angle θ if the numerical range $W(A - \gamma I)$ is contained in S_θ .

In [14] the authors have given a collection of sectorial Operators. Also in this thesis we prove that if $\omega \in [2, \infty)$ and

$$W(A) \cap \{z : \; z < 0\} = \emptyset$$

then A has a ω-sectorial root with vertex 0 and half-angle $\frac{\pi}{\omega}$. Indeed, for every $\omega \in [2, \infty)$ there exists B_ω-the ω-th root of A- such that $W(B_\omega)$ lies in the sector

$$S_\omega = \{z = re^{i\theta} : r \geq 0, \quad |\theta| \leq \frac{\pi}{\omega}\}.$$

1.3 The essential numerical range

The notion of essential numerical range was introduced by Stampfli and Williams in [50]. The essential numerical range, $W_e(A)$, is defined by

$$W_e(A) := \bigcap_{K \in K(\mathcal{H})} \overline{W(T + K)},$$

where $K(\mathcal{H})$ is the ideal of all compact operators on $\mathcal{H}$. We have the following properties on $W_e(A)$ (See [50] for example):

Proposition 1.3.1 *Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$, then:*

1. $W_e(A)$ is a non-void compact and convex set;

2. $W_e(A)$ is the set of points w for which there exists a weakly null sequence $\{x_n\}$ of unit vectors such that $< Ax_n, x_n > \to w$ as $n \to \infty$;

3. $W_e(A) \subseteq \overline{W(A)}$, the closure of $W(A)$;

4. $\overline{W(A)} = co(W_e(A) \cup W(A))$, and so $W(A)$ is closed if and only if $W_e(A) \subseteq W(A)$;

5. $W_e(A) = \{0\}$ if and only if A is compact. Thus for compact operator K, $W(K)$ is closed if and only if $0 \in W(K)$.

1.4 The numerical range attainment problem

In this section, we consider one of the basic problems in the theory of the numerical range: the numerical range attainment problem. The problem asks for a characterization of numerical range of Hilbert space operators. In other words, it asks to determine which nonempty bounded convex subset of the complex plane is the numerical range of some Hilbert space operator. If no restriction on the (size of the) Hilbert space is imposed, then the problem has an easy answer(cf. [40]):

If the space has dimension $\geq 2^{\aleph}$ then every bounded convex subset of the plane is the numerical range of a normal operator. Indeed, let Δ be a bounded convex non-empty subset of the plane. If Δ consists of exactly one point, say

$\Delta = \{\lambda_0\}$, then $W(\lambda_0 I) = \Delta$. If Δ has more than one point then since it is convex it has precisely $2^{\aleph}$ points. Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis of cardinality $2^{\aleph}$ and let this basis be indexed by the points of Δ . Define an operator A on $\mathcal{H}$ by $Ae_\lambda = \lambda e_\lambda$ for all $\lambda \in \Delta$. It is easy to check that A is normal and $W(A) = \Delta$.

In light of this, the attainment problem should be modified in order to remain to be interesting. The modified question asks: which nonempty bounded convex subset of $\mathbb{C}$ is the numerical range of some operator on a separable Hilbert space? One way to approach it is characterize numerical ranges of operators in some special classes. For example, the numerical range of any normal operator on a separable Hilbert space is a Borel set, or for compact operator A, if 0 is in the interior of $W(A)$, then $\partial W(A)$ is the union of finitely many analytic arcs [39]. Also the numerical range of any operator on a separable Hilbert space is $G_{\delta\sigma}$ [8]. Recall that G_δ is the intersection of countably many open set and a $G_{\delta\sigma}$ is the union of countably many G_δ.

Consider the attainment problem for operators in general. We say that a nonempty bounded convex subset of $\mathbb{C}$ is attainable if it is the numerical range of some operator on a separable Hilbert space. Note that non attainable convex sets do exist. Since, as observed in [41], if Δ is a bounded convex open subset of the plane whose closure has uncountably many extreme points, then there exist convex sets with interior Δ which are not attainable (by cardinality reason).

In [41] the authors have shown that if Δ is a bounded convex set in the plane such that $\Delta \backslash \Delta^o$ is a countable union of arcs of conic sections and singletons, then Δ is the numerical range of an operator on a separable Hilbert space. In the end of [41] authors asked, is every compact convex set the numerical range of some operator on separable Hilbert space. In this section we answer, in particular, this question.

Suppose $\mathcal{H}$ is a Hilbert space and A a bounded linear operator on $\mathcal{H}$. A closed linear subspace M of $\mathcal{H}$ is called reducing subspace for A if $A(M) \subseteq M$ and $A(M^{\perp}) \subseteq M^{\perp}$. A vector e_0 in $\mathcal{H}$ is a star-cyclic vector if $\mathcal{H}$ is the smallest reducing subspace for A that contains e_0. The operator A is star cyclic if it has a star-cyclic vector. We will make use of the following proposition which can be found in [15]

Proposition 1.4.1 *A vector e_0 is a star-cyclic vector for A if and only if $\mathcal{H} = \overline{\{Se_0 : S \in C^*(A)\}}$, where $C^*(A)$ is the C^*-algebra generated by A.*

Proposition 1.4.2 *If A has a star-cyclic vector, then $\mathcal{H}$ is separable.*

Theorem 1.4.3 *Any compact convex subset of the plane is the closure of numerical range of a normal operator on a separable Hilbert space.*

Proof. Let K be a compact convex subset of the plane and μ be a regular Borel measure with support K. Define N_μ on $L^2(\mu)$ by $N_\mu f = zf$ for each$f$ in $L^2(\mu)$. It is easy to check that N_μ is normal and $\sigma(N_\mu) = K$. On the other hand $C^*(N_\mu) = \{M_u : u \in C(K)\}$, where M_u denote the multiplication operator by u on $L^2(\mu)$ and $C(K)$ is the continuous complex function on K. Then $\{S1 : S \in C^*(N_\mu)\} = \{M_u(1) : u \in C(K)\} = C(K)$, and $C(K)$ is dense in $L^2(\mu)$, so N_μ is star cyclic operator by proposition 2.3.1 and by proposition 1.4.2 $L^2(\mu)$ is separable. Now the proof is completed because the numerical range of a normal operator is the convex hull of its spectrum. □

In the end of this section we conclude some comments.

Corollary 1.4.4 *For $i = 1, 2$, let K_i are compact convex subsets of plane and μ_i be regular Borel measures with support K_i. If $N_{\mu_1} \cong N_{\mu_2}$, (unitary equivalent), then supportμ_1 = supportμ_2.*

Proof. Since $W(N_{\mu_1}) = W(N_{\mu_2})$, Theorem 4.1.1 implies that, *support*μ_1 = *support*μ_2.□

Corollary 1.4.5 *Let $\lambda \in \mathbb{C} \backslash K$, then $dist(\lambda, K) = \|(\lambda - N_\mu)^{-1}\|^{-1}$.*

Proof. The equality follow from two inequalities (see [39] and Proposition 3.9 of [15]),

$$dist(\lambda, W(N_\mu) = K) \leq \|(\lambda - N_\mu)^{-1}\|^{-1}$$

and

$$\|(\lambda - N_\mu)^{-1}\|^{-1} \leq dist(\lambda, \sigma(N_\mu) = K)$$

□

Chapter 2

NUMERICAL RANGE OF A COMPOSITION OPERATOR

2 Numerical range of Composition operator

In this chapter we introduce the concept of composition operators on the Hardy space H^2 of the unit disc, and investigate the shape of its numerical range. Finally we study the numerical range of finite order elliptic automorphisms composition operators.

2.1 Introduction

Let φ be a holomorphic self-map of the unit disc $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. The function φ induces the *composition operator* C_φ, defined on the space of holomorphic functions on $\mathbb{U}$ by $C_\varphi f = f \circ \varphi$. The restriction of C_φ to various Banach spaces of holomorphic functions on $\mathbb{U}$ has been an active subject of research for more than three decades and it will continue to be for decades to come (see [44], [49] and [16]). Let H^2 denote the *Hardy space* of analytic functions on the open unit disc with square summable Taylor coefficients. In recent years the study of composition operators on the the Hardy space has received considerable attention.

In [38] V. Matache determined the shape $W(C_\varphi)$ when the symbol of the composition operator is a monomial or an inner function fixing 0. Also he gave some properties of the numerical range of composition operators in some cases. In [13] the shapes of the numerical range for composition operators induced on H^2 by some conformal automorphisms of the unit disc specially parabolic and hyperbolic were investigated. In [13], Bourdon and Shapiro have considered the question of when the numerical range of a composition operator is a disc centered at the origin and have shown that this happens whenever the inducing map is a non elliptic conformal automorphism of the unit disc. They also have shown that the numerical range of elliptic automorphism with order 2 is an ellipse with focus at ± 1. In [1], A. Abdollahi has completed their results by finding the exact value of the major axis of the ellipses. However, for the elliptic automorphisms with finite order $k > 2$, this is an open problem yet.

2.2 Notations and Preliminaries

Let $\mathbb{U}$ denote the open unit disc in the complex plane, and the *Hardy space* H^2 the functions $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ holomorphic in $\mathbb{U}$ such that $\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 < \infty$, with $\widehat{f}(n)$ denoting the n-th Taylor coefficient of f. The inner product inducing the norm of H^2 is given by $\langle f, g \rangle := \sum_{n=0}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)}$. The inner product of two functions f and g in H^2 may also be computed by integration:

$$\langle f, g \rangle = \frac{1}{2\pi i} \int_{\partial \mathbb{U}} f(z)\overline{g(z)} \frac{dz}{z}$$

where $\partial \mathbb{U}$ is positively oriented and f and g are defined a.e. on $\partial \mathbb{U}$ via radial limits.

For each holomorphic self map φ of $\mathbb{U}$ induces on H^2 a *composition operator* C_φ, defined by the equation $C_\varphi f = f \circ \varphi (f \in H^2)$. A consequence of a famous theorem of J. E. Littlewood [35] asserts that C_φ is a bounded operator. (see also [44] and [16]). In fact (see [16])

$$\sqrt{\frac{1}{1 - |\varphi(0)|^2}} \leq \|C_\varphi\| \leq \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}.$$