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Some notes about the Lorentzian splitting theorem

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Dedicated to:

My wife and my parents.

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Abstract

In this thesis at first, we introduce some facts about the pseudo-Riemannian geometry and in particular the lorentzian geometry and then, we introduce the Riemannian and lorentzian splitting theorem. Therefore, by consider to these Preliminaries and basic ingredients we obtain the following results.

A lorentzian splitting theorem is obtained for cosmological space-times in a special case and some results about the level sets of Busemann functions are obtained for space- times, and in a special case (cosmological spacetime). These results will be used to prove the conjecture stated by R. Bartnik in [B2], under some special conditions. for this we employ some results of Galloway, Horta and Eschenburg.

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

In 1971, Cheeger and Gromoll [CG1],[CG2] extended the Toponogovs splitting theorem to manifolds of nonnegative Ricci curvature :*A complete Riemannian manifold* (M, g) *of nonnegative sectional curvature which contains a line* γ *must be isometric to* $\mathbb{R} \times S$ *where* $\mathbb{R}$ *is represented by* γ *and* S *is a hypersurface of* M.

In 1982, S.-T. Yau ([Y], p.696) raised the question of showing that *a geodesically complete Lorentzian 4-manifold* M *of nonnegative Ricci curvature in timelike directions which contains a timelike line* γ *must be isometric to* $\mathbb{R} \times S$ *where* $\mathbb{R}$ *is represented by* γ *and* S *is a spacelike hypersurface.* In 1985, Beem et al. [BEMG] obtained the splitting of Lorentzian manifold under the conditions that M be globally hyperbolic and nonpositive timelike sectional curvature with a complete timelike line. In 1988, Eschenburg [E] got the result for M with global hyperbolicity, timelike geodesic completeness and nonnegative Ricci curvature. Galloway [G4] did the same without timelike geodesic completeness. Since the global hyperbolicity is very natural assumption in Lorentzian geometry as geodesic completeness is in Riemannian geometry, those authors always assumed it for the splitting theorems. In fact, since the global hyperbolicity plays very important role in their proofs and since timelike geodesic completeness and the global hyperbolicity are independent in general, it was not easy to replace global hyperbolicity by geodesic completeness. However, in 1990, Newman [N] finally obtained the splitting theorem for M with geodesic completeness, nonnegative Ricci curvature and a distance realizing timelike line . Newmans observation is that the existence of maximal timelike line in Lorentzian theorem is quite strong and most of necessary tools obtained from global hyperbolicity can be obtained from it. Proofs of the above mentioned authors are based on a study of Lorentzian Busemann function.

In this thesis we study some facts about the lorentzian geometry and causality in chapter 2, that be used in the main results which are obtained in the chapter 5 [SB2]. Of course some basic ingredients about the pseudo-Riemannian geometry are stated in the next section.

In chapter 3 and 4 the Lorentzian splitting theorem and the Lorentzian Busemann functions are studied, and some results about the level sets of Lorentzian Busemann functions are obtained. This results help us to prove the new lorentzian splitting theorem for cosmological space-times in chapter 4 [SB1]. By this theorem the following conjecture of R. Bartnik [B2] can be proved by the new additional assumption,

***Conjecture** : if M is a cosmological spacetimes then either M is timlike geodesically incomplete or it splits as a metric product [see sec. 5.1].*

The conjectur should be interpreted as a statement about the rigidity of the Hawking-Penrose singularity theorems: unless spacetime splits (and hence is static), spacetime must be singular, i.e., timelike geodesically incomplete (see [HE]. p. 266).

Several partial proofs of that conjecture rely on the following idea: Construct an inextendible causal geodesic line (which maximizes Lorentzial length on each finite segment), and use a splitting theorem analogous to that of Cheeger and Gromoll in Riemannian geometry (recall that $Ric(X,X) \geq 0$ for all timelike tangent vectors of a cosmological spacetime). The main problem to overcome is that such a line (being constructed as a limit of timelike geodesic segments) might be lightlike rather than timelike which would destroy the argument. Various authors avoided this to happen by introducing additional assumptions. The present work belongs to this series of papers.

In one of the preceding papers [EG] it was assumed that the Cauchy surface S lies in the past of some S-ray (a future directed geodesic ray maximizing distance from each of its points to S). The authors of the this worke assume instead that some horosphere (level set of the Busemann function) of the S-ray is contained in the future Cauchy domain of S (which need to be only an acausal hypersurface). This assumption is weaker than that of [EG] since not all points of S are involved.

1.2 Pseudo-Riemannian Geometry

We begin with a brief introduction to pseudo-Riemmanian geometry.

1.2.1 Manifolds

Let M^n be a smooth n-dimensional manifold. Hence, M^n is a topological space (Hausdorff, second countable), together with a collection of coordinate charts $(U, x) = (U, x^1, \dots, x^n)$ (U open in M) covering M such that on overlapping charts (U, x), (V, y), $U \cap V \neq \emptyset$, the coordinates are smoothly related

$$y^i = f^i(x^1, \dots, x^n), \qquad f^i \in C^\infty, \qquad i = 1, \dots, n.$$

For any $p \in M$, let T_pM denote the tangent space of M at p. Thus, T_pM is the collection of tangent vectors to M at p. Formally, each tangent vector $X \in T_pM$ is a derivation acting on real valued functions f, defined and smooth in a neighborhood of p. Hence, for $X \in T_pM$, $X(f) \in R$ represents the directional deriviative of f at p in the direction X.

If p is in the chart (U, x) then the coordinate vectors based at p,

$$\frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p, \dots, \frac{\partial}{\partial x^n}|_p$$

form a basis for T_pM. I.e., each vector $X \in T_pM$ can be expressed uniquely as,

$$X = X^i \frac{\partial}{\partial x^n}|_p, \qquad X^i \in \mathbb{R}.$$

Here we have used the Einstein summation convention: If, in a coordinate chart, an index appears repeated, once up and once down, then summation over that index is implied.

Note: We will sometimes use the shorthand: $\partial_i = \frac{\partial}{\partial x^i}$.

The tangent bundle of M, denoted TM is, as a set, the collection of all tangent vectors,

$$TM = \bigcup_{p \in M} T_pM.$$

To each vector $V \in TM$, there is a natural way to assign to it $2n$ coordinates,

$$V \sim (x^1, \dots, x^n, V^1, \dots, V^n),$$

where $(x^1, \dots, x^n)$ are the coordinates of the point p at which V is based, and $(V^1, \dots, V^n)$ are the components of V with respect to the coordinate basis vectors $\frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p, \dots, \frac{\partial}{\partial x^n}|_p$. By this correspondence one sees that TM forms in a natural way a smooth manifold of dimension $2n$. Moreover, with respect to this manifold structure, the natural projection map $\pi : TM \to M$, $V_p \mapsto p$, is smooth.

1.2.2 Vector fields

A vector field X on M is an assignment to each $p \in M$ of a vector $X_p \in T_pM$,

$$p \in M \to X_p \in T_pM.$$

If (U, x) is a coordinate chart on M then for each $p \in U$ we have

$$X_p = X^i(p)\frac{\partial}{\partial x^i}|_p.$$

This defines n functions $X^i : U \to \mathbb{R}$, $i = 1, \dots, n$, the components of X on (U, x). If for a set of charts (U, x) covering M the components X^i are smooth ($X^i \in C^\infty(U)$) then we say that X is a smooth vector field.

Let $\mathfrak{X}(M)$ denote the set of smooth vector fields on M. Vector fields can be added pointwise and multiplied by functions; for $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$,

$$(X + Y)_p = X_p + Y_p, \qquad (fX)_p = f(p)X_p.$$

From these operations we see that $\mathfrak{X}(M)$ is a module over $C^\infty(M)$.

Given $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, X acts on f to produce a function $X(f) \in C^\infty(M)$, defined by,

$$X(f)(p) = X_p(f).$$

With respect to a coordinate chart (U, x) , $X(f)$ is given by,

$$X(f) = X^i \frac{\partial f}{\partial x^i}.$$

Thus, a smooth vector field $X \in \mathfrak{X}(M)$ may be viewed as a map

$$X : C^\infty(M) \to C^\infty(M), \qquad f \mapsto X(f)$$

that satisfies,

(1) $X(af + bg) = aX(f) + bX(g) \qquad (a, b \in \mathbb{R})$,

(2) $X(fg) = X(f)g + fX(g)$.

Indeed, these properties completely characterize smooth vector fields.

Given $X, Y \in \mathfrak{X}(M)$, the Lie bracket $[X, Y]$ of X and Y is the vector field defined by

$$[X, Y] : C^\infty(M) \to C^\infty(M), \qquad [X, Y] = XY - YX,$$

i.e.

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

With respect to a coordinate chart, $[X, Y]$ is given by

$$\begin{aligned}[X, Y] &= (X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}) \frac{\partial}{\partial x^j} \\ &= (X(Y^j) - Y(X^j)) \frac{\partial}{\partial x^j}.\end{aligned}$$

It is clear from the definition that the Lie bracket is skew-symmetric,

$$[X, Y] = -[Y, X].$$

1.2.4 Pseudo-Riemannian manifolds

Let V be an n-dimensional vector space over $\mathbb{R}$. A symmetric bilinear form $b : V \times V \to \mathbb{R}$ is

(1) positive definite provided $b(v, v) > 0$ for all $v \neq 0$,

(2) nondegenerate provided for each $v \neq 0$, there exists $w \in V$ such that $b(v, w) \neq 0$ (i.e., the only vector orthogonal to all vectors is the zero vector).

Note: 'Positive definite' implies 'nondegenerate'.

A *scalar product* on V is a nondegenerate symmetric bilinear form $\langle , \rangle : V \times V \to \mathbb{R}$. A scalar product space is a vector space V equipped with a scalar product $\langle , \rangle$. Let V be a scalar product space. An orthonormal basis for V is a basis $e_1, \ldots, e_n$ satisfying,

$$\langle e_i, e_j \rangle = \begin{cases} 0 & i \neq j \\ \pm 1 & i = j, \end{cases}$$

or in terms of the Kronecker delta,

$$\langle e_i, e_j \rangle = \varepsilon_i \delta_{ij} \qquad \text{(no sum)}$$

where $\varepsilon_i = \pm 1$; $i = 1, \ldots, n$.

Note: Every scalar product space $(V, \langle , \rangle)$ admits an orthonormal basis.

The signature of an orthonormal basis is the n-tuple $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. It is customary to order the basis so that the minus signs come first. The index of the scalar product space is the number of minus signs in the signature. It can be shown that the index is well-defined, i.e., does not depend on the choice of basis. The cases of most importance are the case of index 0 and index 1, which lead to Riemannian geometry and Lorentzian geometry, respectively.

Definition 1.2.1. *Let M^n be a smooth manifold. A pseudo-Riemannian metric* $\langle , \rangle$

In addition, the Lie bracket is linear in each slot over the reals, and satisfies,

(1) For all $f, g \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$,

$$[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X.$$

(2) (Jacobi identity) For all $X, Y, Z \in \mathfrak{X}(M)$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

1.2.3 Co-vectors and 1-forms

A co-vector ω at $p \in M$ is a linear functional $\omega : T_pM \to \mathbb{R}$ on the tangent space at p. A 1-form on M is an assignment to each $p \in M$ of a co-vector ω_p at p, $p \to \omega_p$. A 1-form ω is smooth provided for each $X \in \mathfrak{X}(M)$, the function $\omega(X)$, $p \mapsto \omega_p(X_p)$, is smooth. Equivalently, ω is smooth provided for each chart (U, x) in a collection of charts covering M, the function $\omega(\frac{\partial}{\partial x^i})$ is smooth on U, $i = 1, \ldots, n$.

Given $f \in C^\infty(M)$, the differential df is the smooth 1-form defined by

$$df(X) = X(f), \qquad X \in \mathfrak{X}(M).$$

In a coordinate chart (U, x) , df is given by,

$$df = \frac{\partial f}{\partial x^i} dx^i,$$

where dx^i is the differential of the i^{th} coordinate function on U.

Note: At each $p \in U$, $\{dx^1, \ldots, dx^n\}$ is the dual basis to the basis of coordinate vectors $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$.

on a M is a smooth assignment to each $p \in M$ of a scalar product $\langle , \rangle_p$ on T_pM,

$$p \to \langle , \rangle_p : T_pM \times T_pM \to \mathbb{R}.$$

such that the index of $\langle , \rangle_p$ is the same for all p.

By 'smooth assignment' we mean that for all $X, Y \in \mathfrak{X}(M)$, the function $\langle X, Y \rangle$, $p \to \langle X_p, Y_p \rangle_p$, is smooth.

Note: We shall also use the letter g to denote the metric, $g = \langle , \rangle$.

Definition 1.2.2. *A pseudo-Riemannian manifold is a manifold M^n equipped with a pseudo-Riemannian metric $\langle , \rangle$. If $\langle , \rangle$ has index 0 then M is called a Riemannian manifold. If $\langle , \rangle$ has index 1 then M is called a Lorentzian manifold.*

If (U, x) is a coordinate chart then the *metric components* g_{ij} are the functions on U defined by,

$$g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle \qquad i, j = 1, \ldots, n.$$

If X, Y are vectors at some point in U then, by bilinearity,

$$\langle X, Y \rangle = g_{ij} X^i Y^j.$$

Thus, the metric components completely determine the metric on U.

Note: The metric $\langle , \rangle$ is smooth iff for each chart (U, x) , the g_{ij}'s are smooth.

Classically, one displays the metric components as

$$ds^2 = g_{ij} dx^i dx^j.$$

Example 1. Euclidean space $\mathbb{E}^n$ as a Riemannian manifold. We equip $\mathbb{R}^n$ with the *Euclidean metric*. Let $(x^1, \ldots, x^n)$ be Cartesian coordinates on $\mathbb{R}^n$. Then for $X, Y \in T_p\mathbb{R}^n$,

$$X = X^i \frac{\partial}{\partial x^i}|_p$$

$$Y = Y^i \frac{\partial}{\partial x^i}|_p,$$

we have

$$\begin{aligned}\langle X, Y\rangle &= X \cdot Y \\ &= \sum_{i=1}^{n} X^i Y^i \\ &= \delta_{ij} X^i Y^j,\end{aligned}$$

where $\delta_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$ is the Kronecker delta.

Example 2. Minkowski space M^{n+1}. This is the Lorentzian analogue of Euclidean space. We equip $\mathbb{R}^{n+1}$ with the Minkowski metric. Let $(x^0, x^1, \ldots, x^n)$ be Cartesian coordinates on $\mathbb{R}^{n+1}$. Then for $X, Y \in T_p\mathbb{R}^{n+1}$,

$$X = X^i \frac{\partial}{\partial x^i}|_p, \qquad Y = Y^i \frac{\partial}{\partial x^i}|_p,$$

we define,

$$\begin{aligned}\langle X, Y\rangle &= -X^0 Y^0 + \sum_{i=1}^{n} X^i Y^i \\ &= \eta_{ij} X^i Y^j,\end{aligned}$$

where $\eta_{ij} = \varepsilon_i \delta_{ij}$, and $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) = (-1, 1, \ldots, 1)$.

1.2.5 Linear connections

We introduce the notion of covariant differentiation, which formalizes the process of computing the directional derivative of vector fields.

Definition 1.2.3. *A linear connection ∇ on a manifold M is an $\mathbb{R}$-bilinear map,*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

$$(X, Y) \mapsto \nabla_X Y$$

satisfying for all $X, Y \in \mathfrak{X}(M)$, $f \in C^\infty(M)$,

(1) $\nabla_{fX} Y = f \nabla_X Y$,

(2) $\nabla_X fY = X(f)Y + f\nabla_X Y$.

$\nabla_X Y$ is called the *covariant* derivative of Y with respect to X. It can be shown that for any $p \in M$, $\nabla_X Y|_p$ depends only on the values of Y in a neighborhood of p and the value of X just at p. In particular, it makes sense to write $\nabla_X Y|_p$ as $\nabla_{X_p} Y$. This can be thought of as the directional derivative of Y at p in the direction of X_p.

In a coordinate chart (U, x) we introduce the *connection coefficients* Γ^k_{ij}, $1 \leq i, j, k \leq n$ which are smooth functions on U defined by,

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$$

where, recall, $\partial_i = \frac{\partial}{\partial x^i}$.

We can show that with respect to a coordinate chart (U, x) , $\nabla_X Y$ can be expressed as,

$$\nabla_X Y = (X(Y^k) + \Gamma^k_{ij} X^i Y^j)\partial_k, \tag{1.2.1}$$

where X^i, Y^i are the components of X and Y , respectively, with respect to the coordinate basis ∂_i.

Note that this coordinate expression can also be written as,

$$\nabla_X Y = X^i Y^k_{;i} \partial_k$$

where we have introduced the classical notation,

$$Y^k_{;i} = \partial_i Y^k + \Gamma^k_{ij} Y^j.$$

1.2.6 The Levi-Civita connection

Definition 1.2.4. *A linear connection ∇ on M is symmetric provided for all $X, Y \in \mathfrak{X}(M)$,*

$$[X, Y] = \nabla_X Y - \nabla_Y X.$$

Using the coordinate expression (1.2.1) for $\nabla_X Y$, one easily checks that a linear connection ∇ is symmetric iff with respect to each coordinate chart, the connection coefficients satisfy,

$$\frac{D}{dt} = \Gamma^k_{ji}, \qquad \text{for} \quad 1 \leq i, j, k \leq n.$$

Definition 1.2.5. *Let $(M, \langle , \rangle)$ be a pseudo-Riemannian manifold, and let ∇ be a linear connection on M. We say that ∇ is compatible with the metric provided for all $X, Y \in \mathfrak{X}(M)$,*

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle,$$

i.e., the metric product rule holds.

Remark 1.2.6. *The standard linear connection on Euclidean space (and on Minkowski space) is symmetric and compatible with the metric.*

Theorem 1.2.7. *(Fundamental theorem of pseudo-Riemannian geometry). On a pseudo-Riemannian manifold there exists a unique linear connection ∇ that is symmetric and compatible with the metric.*