IN THE NAME OF GOD

WEAKLY REGULAR RINGS

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To:

My parents

and

to all those whom I love.

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Now that another page is turned, because of acknowledgement to God, I am gratefull to the suffering of all those that I am their guideness's own. I appreciate from my dear and honourable professor Mr. Dr. Majid Ershad due to his unsparingly guidnesses and efforts. Also, I appreciate from honourable professors, Mr. Dr. Habib Sharif and Mehdi Hakim Hashemi, who undertook the consultant of this smallest student. At the end, I introduce my endlessness grateful to all of my teachers.

ABESTRACT

WEAKLY REGULAR RINGS

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Regular rings were invented in 1936 by von Neumann. Kaplansky in 1970 made the following conjecture: The ring R is von Neumann regalar if and only if R is semiprime and each prime factor ring of R is von Neumann regular. Snider in 1974 by an example showed that the Kaplansky's conjecture can not be true in general. It is stablished that the necessary and sufficient condition for kaplansky's conjecture to hold is "the union of any chain of semiprime ideals in R is semiprime". Weakly rgular rings were invented by Ramamurti in 1973. It is stablished that the kaplansky's improved conjecture is hold for weakly regular rings.

Reduced P. P. rings were studied by Hattori in 1960, it is stablished that each reduced weakly regular ring is a right and left P.P. ring, but the converse is not true. Goodearl in 1979 proved that the maximal right quotient ring of a strongly regular ring is still strongly reguler. But our arguments involve the existence of a simple non-ore domain that shows that the Goodearl's theorem is not true for weakly regular rings, and also in these rings R_p may not exist for a prime ideal P.

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CHAPTER I

INTRODUCTION

Throughout this dissertation , all rings are associative and have identity unless we state. $J(R)$ and $Z(_RR)$ denote respectively the Jacobson radical and the left singular ideal of a ring R .For any subset X of R let $r(X) = \{a \in R | Xa = 0\}$ and $ann(X) = \{a \in R | Xa = aX = 0\}$, also we denote $N^* = N \cup \{0\}$.

1.1 Categories

Definition 1.1 *A category is a class* $\mathbf{C}$ *of objects (denoted* $A, B, C, \ldots$*) togethere with (i) a class of disjoint sets, denoted hom(A,B), one for each pair of objects in* $\mathbf{C}$*; (an element* f *of hom(A,B) in called a morphism from A to B and is denoted by* $f : A \rightarrow B$*) such that (ii) for each triple (A,B,C) of objects of* $\mathbf{C}$ *a fanction* $hom(B,C) \times hom(A,B) \rightarrow hom(A,C)$ *(for morphisms* $f : A \rightarrow B, g : B \rightarrow C$ *this function is written by* $(g,f) \longmapsto g \circ f$ *and* $g \circ f : A \rightarrow C$ *is called the composite of* f *and* g*); all subject to the two axioms: (1) Associativity. if* $f : A \rightarrow B, g : B \rightarrow C$*,* $h{:}C \rightarrow D$ *are morphisms of* $\mathbf{C}$ *then* $h \circ (g \circ f) = (h \circ g) \circ f$.

(2) Identity. For each object B of $\mathbf{C}$ *there exists a morphism* $1_B : B \rightarrow B$ *such that for any* $f : A \rightarrow B, g : B \rightarrow C$, $1_B o f = f$ *and* $g o 1_B = g$.

Definition 1.2 *A concrete category is a category* $\mathbf{C}$ *togethere with a function* σ *that assigns to each object* A *of* $\mathbf{C}$ *a set* $\sigma(A)$ *(called the underlying set of* A*) in such a way that:*

(i) every morphism $A \to B$ *of* $\mathbf{C}$ *is a function on the underlying sets* $\sigma(A) \to \sigma(B)$;

(ii) The identity morphsim of each object A *of* $\mathbf{C}$ *is the dentity function on the underlying set* $\sigma(A)$ *(iii)Composition of morphisms in* $\mathbf{C}$ *agrees with composition of functions on the underlying sets.*

Example 1.1 *The categroy of groups equipped with the function that assigns to each group its underlying set in the usual sense is a concrete category.*

Definition 1.3 *Let* F *be an object in a concrete category,* X *a nonempty set and* $i : X \to F$ *a map(of sets).* F *is free on the set* X *provided that for any object* A *of* $\mathbf{C}$ *and map(of sets)* $f : X \to A$*, there exists a unique morphism* $\overline{f}$ *of* C *such that* $\overline{f} : F \to A$*, and* $\overline{f}i = f$.

1.2 Modules

Definition 1.4 *A right R-module is an abelian group* M *together with a map* $M \times R \to M$ *written* $(x, r) \mapsto xr$ *such that for all* $x, y \in M$ *and* $r, r_1, r_2 \in R$:

(i) $(x + y)r = xr + yr$

(ii) $x(r_1 + r_2) = xr_1 + xr_2$

(iii) $x(r_1 r_2) = (xr_1)r_2$

if R *has an identity elemant* 1_R *and*

iv) $x1_R = x$*, then* A *is said to be a unitary right R-module. We denote*

M_R for the right R-module M. Simlarly a left R-module will be defined. Let M be a right R-module. A subset L of M is a submodule of M if L is an additive subgroup of M and $x \in L, r \in R$, implies that $xr \in L$, we denote $L \leq M_R$ for this.

Definition 1.5 *Let M_R and N_R be R-modules. A map $\alpha : M_R \to N_R$ is an R-module homomorphsim if $\alpha(x+y) = \alpha(x) + \alpha(y)$ and $\alpha(xr) = \alpha(x)r$ for $x, y \in M, r \in R$. The homomorphism α is called an R-module monomorphism if $\alpha(x) = \alpha(y)$ implies that $x = y$ for $x, y \in M$ and this is equivalent to $Ker\alpha = \{x \in M | \alpha(x) = 0\} = \{0\}$. α is an R-module epimorphism if $Im\alpha = \{\alpha(x) | x \in M\} = N$ and finally α is an R-module isomorphism if α is both R-module monomorphism and epimorphism.*

Definition 1.6 *If X is a subset of a module ${}_RM$ over a ring R, then the intersection of all submodules of M containing X is called the submodule generated by X. If X is finite and X generates the module B ,then B is said to be finitely generated. If $\{B_i | i \in I\}$ is a family of submodules of M, Then the submodule generated by $X = \bigcup_{i \in I} B_i$ is called the sum of the modules B_i.*

Theorem 1.1 *Let R be a ring ,${}_RM$ an R-module ,X a subset of M , $\{B_i | i \in I\}$ a family of submodules of ${}_RM$ and $x \in M$.Let $Rx = \{rx | r \in R\}$*

(i) Rx is a submodule of M and the map $R \to Rx$ given $r \mapsto rx$ is an R-module epimorphism.

(ii) The submodule C generated by $X = \{x\}$(the cyclic submodule generated by x) is $\{rx + nx | r \in R, n \in Z\}$.If R has an identity and C is unitary , then $C = Rx$.

(iii) The submodule D generated by X is

$$\{\sum_{i=1}^{s} r_i x_i + \sum_{j=1}^{t} n_j y_j | s,t \in N^*, x_i, y_j \in X, r_i \in R, n_j \in Z\}$$

If R has an identity and M is unitary , then

$$D = \{\sum_{i=1}^{s} r_i x_i | s \in N^*, x_i \in X, r_i \in R\}$$

(iv) The sum of the family $\{B_i | i \in I\}$ *consists of all finite sums* $b_{i1} + \cdots + b_{in}$ *with* $b_{ik} \in B_{ik}$.

Proof. See [10,iv.1.5]

Definition 1.7 *A pair of module homomorphisms* $A \xrightarrow{f} B \xrightarrow{g} C$ *is said to be exact at B provided that* $Imf = Kerg$*.So the sequence* $0 \longrightarrow A \xrightarrow{f} B$ *is exact if and only if f is an R-module monomorphism.*

Let R be a ring , it is clear that R is both right and left R-module.

Definition 1.8 *Any right(left) R-submodule of R is called a right(left) ideal of R.*

Theorem 1.2 *Let A be a left module over an integral domain R and for each* $a \in A$ *let* $O_a = \{r \in R | ra = 0\}$*, then*
(i) O_a *is an ideal of R for each* $a \in A$.
(ii) $A_t = \{a \in A | O_a \neq 0\}$ *is a submodule of A.*

proof :Obvious

The submodule A_t is called the torsion submodule of A .A is said to be a torsion module if $A = A_t$ and to be torsionfree if $A_t = 0$.

1.3 Free modules

Definition 1.9 *A subset X of a left R-module M is said to be linearly independent provided for distinct $x_1, \cdots, x_n \in X$ and $r_i \in R, r_1x_1 + \cdots + r_nx_n = 0$ implies that $r_i = 0$ for every i.If M is generated as an R-module by a set Y then we say that Y spans M.If R has an identity and M is unitary,then Y spans M if and only if every element of M can be written as a linear combination $r_1y_1 + r_2y_2 + \cdots + r_ny_n (r_i \in R, y_i \in Y)$.A linearly independent subset of M that spans M is called a basis of M.*

Theorem 1.3 *Let R be a ring with identity , the following conditions on a unitary left R-module F are equivalent :*

(i) F has a nonempty basis

(ii) F is the internal direct sum of a family of cyclic modules , each of which is isomorphic as a left R-module to R.

(iii) F is R-module isomorphic to a direct sum of copies of the left R-module R.

(iv) There exists a nonempty set X and a function $i : X \longrightarrow F$ with the following property :

Given any unitary R-module M and function $F : X \longrightarrow M$, there is a unique R-module homomorphism $\overline{f} : F \longrightarrow M$ such that $\overline{f}i = f$.In other words ,F is a free object in the category of unitary R-modules.

Proof. See[10,iv.2.1]

Definition 1.10 *A unitary module F over a ring R with identity which satisfies the equivalent conditions of the above theorem ,is called a free R-module.*

1.4 Projective and injective modules

Definition 1.11 *Let R be a ring . A right R-module P is called projective if given any diagram of R-module homomorphisms*

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow f & & \\ A & \xrightarrow{g} & B & \rightarrow & 0 \end{array}$$

with A and B are right R-modules and the bottom row is exact (that is g is an epimorphism),there exists an R-module homomorphism $f : P \rightarrow A$ such that the diagram

$$\begin{array}{ccccc} & & P & & \\ & h\swarrow & \downarrow f & & \\ A & \xrightarrow{g} & B & \rightarrow & 0 \end{array}$$

is commutative(that is $gh = f$).

Proposition 1.1 *Every free right module F over a ring R with identity is projective .*

Proof. See[10,3.2]

Theorem 1.4 *Let R be a ring,the following conditions on a right R-module P are equivalent : (i) P is projective.*

(ii)P is a direct summand of a free right R-module.

Proof. See[17,I.6.1]

Definition 1.12 *A right R-module E is called injective , if given any diagram of R-module homomorphism*

$$\begin{array}{ccccc} 0 & \rightarrow & A & \xrightarrow{g} & B \\ & & \downarrow f & & \\ & & E & & \end{array}$$

which A and B are right R-modules and the top row is exact(that is g is a monomorphism) ,there exists an R-module homomorphism $h : B \to E$ such that the diagram

$$\begin{array}{ccccc} 0 & \to & A & \xrightarrow{g} & B \\ & & \downarrow f & \swarrow h & \\ & & E & & \end{array}$$

is commutative(that is $hg = f$).

Proposition 1.2 *A direct product of R-modules $\prod_{i \in I} J_i$ is injective if and only if J_i is injective for every $i \in I$.*

Proof. See[10,IV.3.7]

Proposition 1.3 *Every unitary module A over a ring R with identity may be embedded in an injective R-module.*

Proof. See[9,4.4]

1.5 Tensor products

Definition 1.13 *Let G be an abelian group.A map $i : L \times M \to G$ where L is a right R-module and M is a left R-module ,will be called bilinear if*

(i) $i(x + x', y) = i(x, y) + i(x', y)$

(ii) $i(x, y + y') = i(x, y) + i(x, y')$

(iii) $i(xr, y) = i(x, ry)$ *for each* $r \in R, x, x' \in L$ *and* $y, y' \in M$.

Definition 1.14 *A tensor product of L_R and $_RM$ is an abelian group T together with a bilinear map $i : L \times M \to T$ such that for every abelian* *group G and bilinear map $j : L \times M \to G$, there exists a unique homomorphism $\alpha : T \to G$ such that the diagram*

$$\begin{array}{ccc} & & T \\ & i \nearrow & \downarrow \alpha \\ L \times M & \xrightarrow{j} & G \end{array}$$

is commutative($\alpha i = j$).

Proposition 1.4 *If $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an exact sequence of left modules over a ring R and D is a right R-module , then*

$$D \bigotimes_R A \xrightarrow{1_D \otimes f} D \bigotimes_R B \xrightarrow{1_D \otimes g} D \bigotimes_R C \longrightarrow 0$$

is an exact sequence of abelian groups.An analogous statement holds for an exact sequence in the first variable.

Proof. See[1,IV.5.4]

Proposition 1.5 *If R is a ring with identity and I is a right ideal of R and M a left R-module ,then*

$$R/I \bigotimes_R M \cong M/IM$$

.

Proof. Consider the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

then

$$I \bigotimes_R M \xrightarrow{\alpha} R \bigotimes_R M \xrightarrow{\beta} R/I \bigotimes M \longrightarrow 0$$

is exact so β is epimorphism .Therefore

$$R/I \bigotimes_R M \cong M/Ker\beta = M/Im\alpha = M/IM$$

.

Definition 1.15 *A left R-module M is called flat if for each exact sequence*

$$0 \longrightarrow N_1 \xrightarrow{\alpha} N_2 \xrightarrow{\beta} N_3 \longrightarrow 0$$

with $N_1, N_2, N_3$are right R-modules ,the sequence

$$0 \longrightarrow N_1 \bigotimes_R M \xrightarrow{\alpha \otimes 1_M} N_2 \bigotimes_R M \xrightarrow{\beta \otimes 1_M} N_3 \bigotimes_R M \longrightarrow 0$$

is exact.

1.6 Chain conditions

A module A is said to satisfy the ascending chain condition(ACC) on submodules (or to be Noetherian) if for every chain $A_1 \subset A_2 \subset A_3 \subset \cdots$of submodules of A , there is an integer n such that $A_i = A_n$for all $i \geq n$. A module B is said to satisfy the discending chain condition(DCC) on submodules(or to be Artinian) if for every chain $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$ of submodules of B there is an integer m such that $B_i = B_m$for all $i \geq m$.A ring R is left(right) Noetherian if it satisfies the ascending chain condition