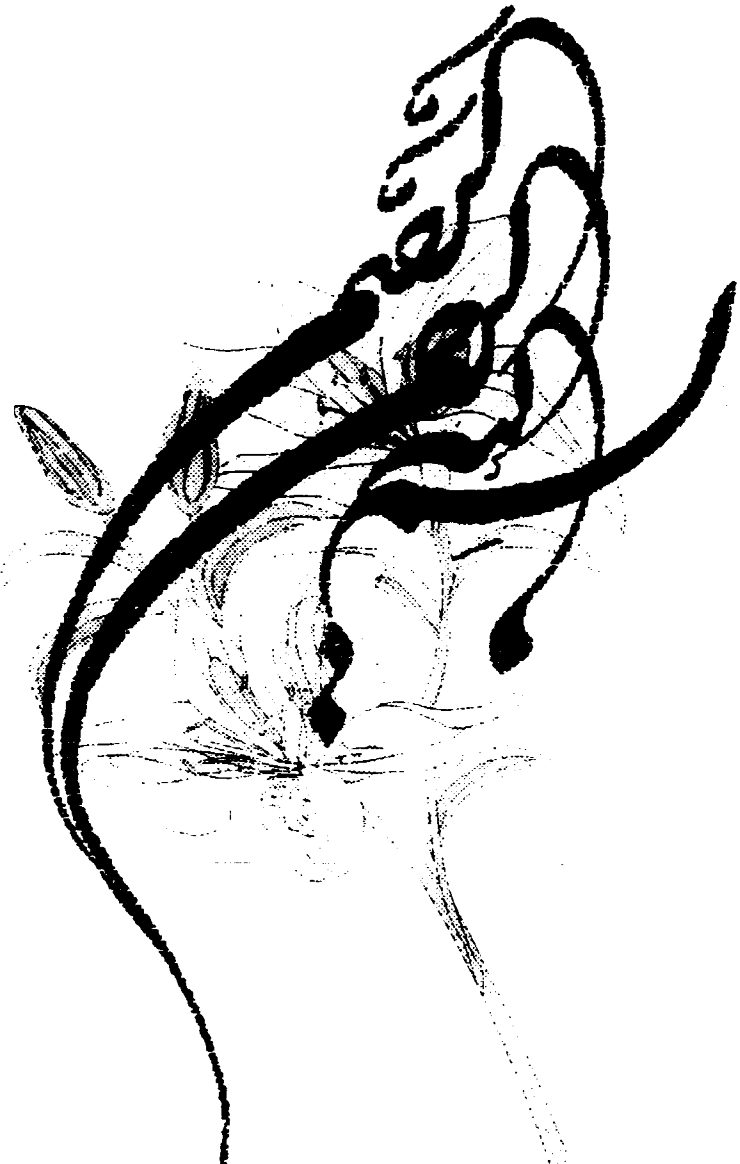

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AMENABILITY OF BANACH ALGEBRAS

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A thesis submitted to Ferdowsi university of Mashhad for the degree of the Doctor of philosophy in Mathematics

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شماره ......
تاریخ ....
پیوست ......

دانشکدهٔ علوم ریاضی

دانشگاه فردوسی مشهد

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Preface

"People do acquire a little brief authority by equipping themselves with jargon : they can pontificate and air a superfical expertise. But what we should ask of educated mathematicians is not what they can speechify about, nor even what they know about the existing corpus of mathematical knowledge, but rather what can they now do with their learning and whether they can actually solve mathematical problems arising in practice. In short, we look for deeds not words." J.Hammersley [29].

"Mathematics is a tool for tought. A highly necessary tool in a word where both feedback and nonlinearities abound. Similary, all kinds of parts of mathematics serve as tools for other parts and for other sciences." A. Helemskii [31].

The subject of amenability essentially begins with Lebesgue (1904). One of the properties of his integral is a version of monotone convergence theorem, and Lebegue asked if this property was really fundamental, that is, if the property follows from the more familar integral axioms. Now the monotone convergence theorem is equivalent to countable additivity, and so the question is concerned with the existence of a positive, finitely (but not countably) additive, translation invariant measure μ on R with $\mu([0,1]) = 1$.

The classical period (1904-1938) is therefore concerned with the study of finitely additive, invariant measure theory.

The modern period begins in the 1940 and continues with increasing energy to the present. The main shift is from finitely additive measures to means: integrating a positive, finitely additive measure μ on a set X with $\mu(X) = 1$ gives rise to mean m on X, that is a connection linear functional on $\ell^\infty(X)$ such that $m(1) = 1 = \|m\|$. The connection between μ and m is given by $\mu(E) = m(\chi_E)$, and the correspondence $\mu \to m$ is bijective. Hence a group G is called amenable if and only if there exists a left invariant means on G.

In the 1940 and 1950, the subject of amenable groups and amenable semigroups was studied by M. M. Day, and his 1957 paper on amenable semigroups is a major landmark.

The ubiquity of amenability ideas and the depth of the mathematics with which the subject is involved seems evidence to the author that here we have a topic of fundamental importance in modern mathematics, one that deserves to be more widely known that it is at present. A good example of how amenability ideas can cross mathematical categories is afforded by the theory of amenable Banach algebras. This theory, which is largely the creation of B. E. Johnson, is discussed in chapter 2. A Banach algebra A is called amenable if the first cohomology group, $H(A, X^*) = 0$ for every dual Banach A-module X^*. This means that every continuous derivation $D : A \to X^*$ is inner. At first sight, there does not seem to be any connection between invariant means and derivations. However, Johnson proved the remarkable result that if G is a locally compact group, then G is amenable if and only if the Banach algebra $L^1(G)$ is amenable. This justifies the terminology amenable Banach algebra.

The other works that has recently appeared (1991), relevant to the relationship between amenability of Banach algebras and amenability of groups (semigroups) are the papers [45], [46], [47], by A. L. Paterson. He proved that a C^*-algebra A is amenable if and only if $U(A)$, the unitary group of A, is amenable. We have attempted to describe the main lines and developed this subject.

(Chapters 1 and 2 establish the basic theory of amenability of topological groups and amenability of Banach algebras. Also we prove that

If G is a topological group, then $\mathcal{R}(WLUC(G)) \neq \emptyset$ [*resp.* $\mathcal{R}(LUC(G)) \neq \emptyset$] if and only if there exists a mean m on $WLUC(G)$ [*resp.* $LUC(G)$] such that for every $f \in WLUC(G)$ [resp. every $f \in LUC(G)$] and every element d of a dense subset D of G, $m(R_d f) = m(f)$ holds.

Chapter 3 investigates relations between amenability of Banach algebras and groups (semigroups). We show that a A^*-algebra A is amenable if $U(A)$, the unitary group of A, is amenable. Furthermore, give an example that the converse is not true in general. Also we prove that

If G is a bounded subset of a unital Banach algebra A such that G is a group w.r.t. multiplication operation of A, and $\overline{span}(G) = A$. If G is a topological group w.r.t. $\sigma(A, A^*)$-topology and G is amenable, then A is an amenable Banach algebra. Thus, we show that:

The following statements are equivalent for a von Neumann algebra M, with unitary group H and isometry semigroup I.

(1) M is injective.

(2) there exists a right invariant mean on $WLUC(H)$.

(3) there exists a right invariant mean on $WLUC(I)$.

Also, the following statements are equivalent for a C^*-algebra A, with unitary group G and isometry semigroup S.

(1) A is nuclear.

(2) there exists a right invariant mean on $LUC(G)$.

(3) there exists a right invariant mean on $LUC(S)$.

Throughout the thesis we use results from certain standard textbooks. These textbooks are listed, together with other references, at the Bibliography. I am very grateful to my supervisor professor A. Niknam for mathematical discussions and advice.

Contents

Chapter 3. Relations between Amenable Banach algebras and its groups

1.Banach algebras

2.von Neumann algebras

3.C^*-algebras

4.A^*-algebras

Chapter 1

Amenable topological groups

This chapter establish the basic theory of means on various functions spaces and amenability of topological groups. Also we prove that

If G is a topological group, then $\mathcal{R}(WLUC(G)) \neq \emptyset$ [*resp.* $\mathcal{R}(LUC(G)) \neq \emptyset$] if and only if there exists a mean m on $WLUC(G)$ [*resp.* $LUC(G)$] such that for every $f \in WLUC(G)$ [resp. every $f \in LUC(G)$] and every element d of a dense subset D of G, $m(R_d f) = m(f)$ holds.

This result are used in chapter 3, when we investigate relationship between amenability of a Banach algebra A, and existence invariant means on $WLUC(G)$, where G is the unitary group of A.

1 Means on various functions spaces

Throughout this section S denotes an arbitrary nonempty set. Recall that $\ell^\infty(S)$ is the C^*-algebra of all bounded complex-valued functions on S.

1.1 Definition

Let $\mathcal{F}$ be a linear subspace of $\ell^\infty(S)$ and let $\mathcal{F}_r$ denote the set of all real-valued members of $\mathcal{F}$. A mean on $\mathcal{F}$ is a linear function μ on $\mathcal{F}$ with the property that

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s) \qquad (f \in \mathcal{F}_r) \qquad (1)$$

The set of all means on $\mathcal{F}$ is denoted by $\mathcal{M}(\mathcal{F})$. If $\mathcal{F}$ is also an algebra and if $\mu \in \mathcal{M}(\mathcal{F})$ satisfies

$$\mu(fg) = \mu(f)\mu(g) \qquad (f, g \in \mathcal{F})$$

then μ is said to be multiplicative. The set of all multiplication means on $\mathcal{F}$ is called the spectrum of $\mathcal{F}$ and will be denoted by $\mathcal{MM}(\mathcal{F})$.

1.2 Proposition

Let $\mathcal{F}$ be a conjugate closed linear subspace of $\ell^\infty(S)$ containing the constant functions. A mean μ on $\mathcal{F}$ has the following properties.

(i) μ is positive, that is, if $f \in \mathcal{F}_r$ and $f \geq 0$, then $\mu(f) \geq 0$.

(ii) $\mu(1) = 1$.

(iii) μ is a bounded linear functional on $\mathcal{F}$ with $\|\mu\| = 1$.

(iv)For all $f \in \mathcal{F}, \mu(\mathcal{R}e\ f) = \mathcal{R}e(\mu(f)), \mu(\mathcal{I}m\ f) = \mathcal{I}m(\mu(f))$ and $\mu(\overline{f}) = \overline{\mu(f)}$.

(v) $\mu(f)$ is in the closed convex hull of $f(S)$ for all $f \in \mathcal{F}$.

Conversely, a linear functional μ on $\mathcal{F}$ that satisfies any two of properties (i), (ii), (iii) is a mean. ([4], Chapter 2, Proposition 1.2)

1.3 Definition

Let $\mathcal{F}$ be a conjugate closed linear subspace of $\ell^\infty(S)$ containing the constant functions. For each $s \in S$ define $\epsilon(s) \in \mathcal{M}(\mathcal{F})$ by

$$\epsilon(s)(f) = f(s) \qquad (f \in \mathcal{F})$$

The mapping $\epsilon : S \to \mathcal{M}(\mathcal{F})$ is called the evaluation mapping, and $\epsilon(s)$ is called evaluation at s. If $\mathcal{F}$ is also an algebra, then $\epsilon(S) \subseteq \mathcal{MM}(\mathcal{F})$, hence we may write $\epsilon : S \to \mathcal{MM}(\mathcal{F})$. In the setting being developed here, the natural topology of $X = \mathcal{M}(\mathcal{F})$ or $X = \mathcal{MM}(\mathcal{F})$ is the relative weak*-topology $\sigma(X, \mathcal{F})$.

1.4 Definition

Let $\mathcal{F}$ be a conjugate closed linear subspace (respectively, subalgebra) of $\ell^\infty(S)$ containing the constant functions, and let $X = \mathcal{M}(\mathcal{F})$ (respectively, $X = \mathcal{MM}(\mathcal{F})$) be furnished with the relative weak*-topology. For each $f \in \mathcal{F}$ the function $\hat{f} \in C(X)$ is defined by

$$\hat{f}(\mu) := \mu(f) \qquad (\mu \in X).$$

further, we define

$$\hat{\mathcal{F}} := \{\hat{f} : f \in \mathcal{F}\}.$$

1.5 Remark

The mapping $f \to \hat{f} : \mathcal{F} \to C(X)$ is clearly linear (and multiplicative if $\mathcal{F}$ is an algebra and $X = \mathcal{MM}(\mathcal{F})$), preserves complex conjugation, and is an isometry since for any $f \in \mathcal{F}$

$$\|\hat{f}\| = \sup\{|\mu(f)| : \mu \in X\} \leq \sup\{|\mu(f)| : \mu \in C(X)^*, \|\mu\| \leq 1\} = \|f\|$$

$$= \sup\{|f(s)| : s \in S\} = \sup\{|\hat{f}(\epsilon(s))| : s \in S\} \leq \|\hat{f}\|$$

where $\epsilon : S \to X$ denotes the evaluation mapping, so that

$$\hat{f}(\epsilon(s)) = f(s) \qquad (f \in \mathcal{F}, s \in S).$$

This last identity may be written in terms of the dual map $\epsilon^* : C(X) \to \ell^\infty(S)$ as $\epsilon^*(\hat{f}) = f \quad (f \in \mathcal{F})$. The topological and geometric structure of $\mathcal{M}(\mathcal{F})$ is described in the following theorem.

1.6 Theorem

Let $\mathcal{F}$ be a conjugate closed linear subspace of $\ell^\infty(S)$ containing the constant functions. Then the following assertions hold.

(i) $\mathcal{M}(\mathcal{F})$ is convex and weak* compact.

(ii) $co(\epsilon(S))$ is weak* dense in $\mathcal{M}(\mathcal{F})$.

(iii) $\mathcal{F}^*$ is the weak* closed linear span of $\epsilon(S)$.

(iv) If $\mathcal{F}$ is also an algebra, then $\mathcal{MM}(\mathcal{F})$ is weak* compact and $\epsilon(S)$ is weak* dense in $\mathcal{MM}(\mathcal{F})$.

(v) If S is a topological space and $\mathcal{F} \subseteq C(S)$, then $\epsilon : S \to \mathcal{M}(\mathcal{F})$ is weak* continuous. ([4], Chapter 2, Theorem 1.8)

2 Invariant means

Throughout this section S denotes an arbitrary semigroup. A central problem in the theory of means is to determine whether or not a given space of bounded functions on semigroups possesses a mean that is left (or right) translation invariant.

2.1 Definition

Let $f \in \ell^\infty(S)$ and let $s \in S$. The right (respectively, left) translate of f by s is the function $R_s f$ (respectively, $L_s f$) where $R_s f(t) = f(ts)$, $L_s f(t) = f(st)$. A subset $\mathcal{F}$ of $\ell^\infty(S)$ is said to be right (respectively, left) translation invariant if for every $f \in \mathcal{F}$ and every $s \in S$, $R_s f \in \mathcal{F}$ (respectively, $L_s f \in \mathcal{F}$). $\mathcal{F}$ is translation invariant if it is both right and left translation invariant.

Let $\mathcal{F}$ be a left (respectively, right) translation invariant, conjugate closed, linear subspace of $\ell^\infty(S)$ containing the constant functions. A member μ of $\mathcal{F}^*$ is said to be left (respectively, right) invariant if , for all $f \in \mathcal{F}$ and $s \in S$, $\mu(L_s f) = \mu(f)$ (respectively, $\mu(R_s f) = \mu(f)$). The set of all left (respectively, right) invariant means on $\mathcal{F}$ is denoted by $\mathcal{L}(\mathcal{F})$ (respectively, $\mathcal{R}(\mathcal{F})$). $\mathcal{F}$ is said to be left (respectively, right) amenable if $\mathcal{L}(\mathcal{F}) \neq \emptyset$ (respectively, $\mathcal{R}(\mathcal{F}) \neq \emptyset$). If $\mathcal{F}$ is translation invariant we set

$$\mathcal{I}(\mathcal{F}) := \mathcal{L}(\mathcal{F}) \cap \mathcal{R}(\mathcal{F})$$

and call members of $\mathcal{I}(\mathcal{F})$ invariant means. $\mathcal{F}$ is said to be amenable if $\mathcal{I}(\mathcal{F}) \neq \emptyset$.

S is said to be left amenable, right amenable, or amenable if the appropriate property holds for $\ell^\infty(S)$.

2.2 Remark

It is easy to check that $\mathcal{L}(\mathcal{F})$, when nonempty, is a weak* closed convex subset of $\mathcal{F}^*$. The same holds for $\mathcal{R}(\mathcal{F})$. It follows that if $\mathcal{F}$ has two distinct left (or right) invariant means, then it has infinitely many.

Let S be a group and let $\mathcal{F}$ be a linear subspace of $\ell^\infty(S)$. For each $f \in \mathcal{F}$ define $\tilde{f} : S \to \mathcal{C}$ by $\tilde{f}(s) := f(s^{-1}) \qquad (s \in S)$,

and set $\tilde{\mathcal{F}} := \{\tilde{f} : f \in \mathcal{F}\}$ if $\mu \in \mathcal{F}^*$, define $\tilde{\mu} \in \tilde{\mathcal{F}}^*$ by

$$\tilde{\mu}(\tilde{f}) = \mu(f) \qquad (f \in \mathcal{F})$$

If $\tilde{\mathcal{F}} = \mathcal{F}$ and $\tilde{\mu} = \mu$ then μ is said to be inversion invariant.

2.3 Example

(i) Let $S = \{s_1, s_2, \cdots, s_n\}$ be a finite left cancellative semigroup.

Then $n^{-1}\sum_{i=1}^{n} \epsilon(s_i)$ is a left invariant mean on $\ell^\infty(S)$, where ϵ is the evaluation mapping.

(ii) If S is a compact, Hausdorff, topological group, then $C(S)$ has a unique invariant mean that is also inversion invariant. ([4], Chapter 2, Corollary 3.12)

(iii) Every abelian semigroup S is amenable. ([4], Chapter 2, Corollary 3.8)