In The Name of God

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Some Properties in Variety of Groups

by

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In the Name of God

the Beneficent , the Merciful

To My Parents ,

To My Wife ,

To My Children

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Introduction

This thesis is a partial fulfilment of the Ph.D. degree, submitted to Ferdowsi University of Mashhad. I should like to express my sincere thanks and gratitute to my supervisor Prof. M.R.R. Moghaddam, for his advise and guidence throughout writting my thesis.

The thesis has been arranged into four chapters and mainly concerned with the Baer-invariant of groups which is the generalization of the Schur-multiplier with respect to any variety of groups. In fact we have tried to find some relationships between the Baer-invariants of groups, verbal subgroup, marginal subgroup, and some group theoretical concepts.

To be more precise, the contexts of each chapter are designed as follows:

Chapter One is devoted to collect some notion and background informations, which are needed in the next chapters. It also contains some important statements which will be proved in a more general context later in this thesis.

In chapter Two, we show that if the marginal factor-group is of order $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}, n \geq 1$, then we obtain a bound for the order of the verbal subgroup. Also a bound for the Baer-invariant of a finite p-group with respect to the variety of polynilpotent groups of a given class row will be constructed.

Chapter Three is devoted to present some inequalities for the Baer-invariants of a finite group, with respect to a given variety of groups. Using these results a generalized version of a theorem of Stallings (1965) will be proved. It is also given a sufficient condition for a family of $\mathcal{V}$-nilpotent groups, which does not have any $\mathcal{V}$-covering groups, with respect to a certain variety of groups $\mathcal{V}$.

In chapter Four, we study the concepts of $\mathcal{V}$-isologisms and $\mathcal{V}$-marginal extensions of groups, with respect to a given variety of groups $\mathcal{V}$. Finally we give equivalent conditions under which two extensions are $\mathcal{V}$-isologic.

Chapter One

Notation and Background Informations

In preliminary chapter we introduce and collect all the necessary notations and results, which will be needed throughout this thesis. Especially, we discuss about commutator identities, second cohomology of groups, varieties of groups, Baer-invariants, Schur-Baer property, covering groups, isologism properties and marginal extensions.

We also state some of the known results explicitly,without proofs which are needed in the following chapters.

" Commutators and their Properties "

Here we provide the definitions of simple, outer and basic commutators, and state some of their important properties, which will be used in the comming chapters.

Let x and y be two elements of a group G, then $[x, y]$, the *commutator* of x and y, and x^y denote the elements $x^{-1}y^{-1}xy$ and $y^{-1}xy$, respectively. The *commutator* of *higher weight* is defined inductively as follows:

$$[x_1, x_2, \ldots, x_{n-1}, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n] \quad (n > 2)$$

Then $[x_1, x_2, \ldots, x_{n-1}, x_n]$ is called a *left-normed commutator of weight* n in $x_1, x_2, \ldots, x_{n-1}, x_n$. A right-normed commutator may be defined similarly, but we always work with left-normed commutators. If H and K are two subgroups of a group G, then $[H, K]$ denotes the subgroup of G generated by all the commutators $[h, k]$ with h in H and k in K. In particular, if $H = K = G$ then $[G, G]$, which is denoted by G', is the *derived subgroup* of G. Also if $H_1, H_2, \ldots, H_n$ are subgroups of G, then we define

$$[H_1, H_2, \ldots, H_{n-1}, H_n] = [[H_1, H_2, \ldots, H_{n-1}], H_n] \quad (n > 2)$$

Henceforth, the commutator $[x, y, y, \ldots, y]$ in which y is repeated n times, will be denoted by $[x,_n y]$, and similarly the commutator subgroup $[H, K, K, \ldots, K]$ for n repeatation of K, will also be denoted by $[H,_n K]$. The terms of the lower and upper central series for a group G are denoted by $\gamma_n(G)$; $(n \geq 1)$, and $Z_m(G)$; $(m \geq 0)$, respectively, and are defined as follows;

Definition 1.1

Let G be a group, the *lower central series* $\{\gamma_n(G)\}_{n\geq 1}$ of G is defined to be;

$$\gamma_1(G) = G \ , \ \gamma_2(G) = G' \ , \ \gamma_n(G) = [_{n-1}G, G] \ = \ [\gamma_{n-1}(G), G] \ , \ n \geq 2$$

Therefore we have:

$$\gamma_1(G) \supseteq \gamma_2(G) \supseteq \ldots \gamma_n(G) \supseteq \gamma_{n+1}(G) \supseteq \ldots .$$

The *upper central* series $\{Z_m(G)\}_{m\geq 0}$ of G is also defined as follows:

$$Z_0(G) = 1 \ , \ Z_1(G) = Z(G) \ , \ \frac{Z_m(G)}{Z_{m-1}(G)} = Z(\frac{G}{Z_{m-1}(G)}) \qquad (m \geq 1)$$

in which Z(G) denotes the *center* of the group G. We therefore have

$$Z_0(G) \subseteq Z_1(G) \subseteq Z_2(G) \subseteq \ldots \subseteq Z_m(G) \subseteq Z_{m+1}(G) \subseteq \ldots$$

Definition 1.2

A group G is said to be *nilpotent* if there exists a natural number n, such that $\gamma_{n+1}(G) = 1$. If the natural number c is the least integer such that $\gamma_{c+1}(G) = 1$ (i.e. $\gamma_c(G) \neq 1$), then G is called a *nilpotent group* of class c.

It can be easily checked that a group G is nilpotent of class at most c if and only if $Z_c(G) = G$.

Definition 1.3

A group G is said to be *soluble* (or *solvable*) if it has an abelian series, by which we mean a finite series $1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_n = G$ in which each factor G_{i+1}/G_i is abelian, for $0 \leq i < n$.

Lemma 1.4

(i) Every abelian group is nilpotent and every nilpotent group is soluble.

(ii) Let G be a finite group. Then G is a nilpotent group if and only if it is the direct product of its sylow subgroups.

Proof. It is clear,(see also [39] or [6])

The following commutator identities are used throughout this work frequently.

Lemma 1.5

Let G be any group, then for all $x, y, z \in G$, we have:

(1) $[x,y]^{-1}=[y,x]$, $x^y=x[x,y]$

(2) $[x^{-1},y]=[y,x]^{x^{-1}}$, $[x,y^{-1}]=[y,x]^{y^{-1}}$

(3) $[xy,z]=[x,z]^y[y,z]=[x,z][x,z,y][y,z]$ (Hall's identity)

(4) $[x,yz]=[x,z][x,y]^z=[x,z][x,y][x,y,z]$ (Hall's identity)

(5) $[y,x,z^y][z,y,x^z][x,z,y^x]=1$ (Witt's identity)

(6) $[x,y^{-1},z]^y[y,z^{-1},x]^z[z,x^{-1},y]^x=1$

(7) If $z=[x,y]$ commutes with both x and y, then

(i) $[x^i,y^j]=z^{ij}$ for all $i,j\in N$

(ii) $(yx)^i=z^{i(i-1)/2}y^ix^i$ for all $i\in N$

(iii) $[x,y]^{-1}=[x^{-1},y]=[x,y^{-1}]$

(8) If $y\in C_G(z)$ and $[x,G]$ is abelian, then $[x,y,z]=[x,z,y]$

(9) For all $f_1,f_2,\dots,f_n\in G$, then module $\gamma_{n+1}(G)$,

$$[f_1,\dots,f_{i-1},f_i,\dots,f_n]=[f_1,\dots,f_i,f_{i-1},\dots,f_n]$$

$$[f_1,\dots,f_{i-2},[f_{i-1},f_i],f_{i+1},\dots,f_n],$$

(10) If $y\in\gamma_r(G)$, then for any natural number n we have

(i) $[x^n,y]\equiv[x,y^n]\equiv[x,y]^n \quad (mod\ \gamma_{r+2}(G))$

(ii) $(yx)^n=x^ny^n[x,y]^{\binom{n}{2}} \quad (mod\ \gamma_{r+2}(G))$

Proof. The proofs of some parts can be founded in [39],[6]. Others are strightforward using the commutator manipulations.

Lemma 1.6

(1) Let X and Y be two subsets of a group G and K be a subgroup of it, then

(i) $< X >^K = < X, [X, K] >$.

(ii) $[X, K]^K = [X, K]$.

(iii) If $K =< Y >$, then $[X, K] = [X, Y]^K$.

(iv) If $K =< Y >$ and $H =< X >$, then $[H, K] = [X, Y]^{HK}$.

(2) H normalizes K (i.e. $H \trianglelefteq N_G(K)$) if and only if $[H, K] \subseteq K$.

(3) If $K \trianglelefteq G$, then G/K is abelian if and only if $[G, G] \subseteq K$.

(4) If $K \subseteq H$ and $H, K \trianglelefteq G$, then

$$H/K \subseteq Z(G/K) \iff [H, G] \subseteq K$$

(5) If $H, K, L \trianglelefteq G$, then $[HK, L] = [H, L][K, L]$ and $[H, K] \trianglelefteq G$.

(6) (Three subgroup lemma)

If $H, K, L \leq G$ and $N \trianglelefteq G$ and $[H, K, L], [K, L, H] \subseteq N$, then $[L, H, K] \subseteq N$.

(7) $[\gamma_i(G), \gamma_j(G)] \leq \gamma_{i+j}(G)$, for all $i, j \in Z^+$,

(8) $\gamma_i(\gamma_j(G)) \leq \gamma_{ij}(G)$,

(9) $[\gamma_i(G), Z_j(G)] \leq Z_{j-i}(G)$, $(i \leq j)$,

(10) $Z_i(G/Z_j(G)) = Z_{i+j}(G)/Z_j(G)$,

(11) $G^{(i)} \leq \gamma_{2^i}(G)$, where $G^{(i)}$ is the i-th derived subgroup of G.

Proof. See [39] or [6].

Definition 1.7

Let F_∞ be a free group with a set of generators $\{x_1, x_2, \ldots, x_n, \ldots\}$, then *the outer commutators* are defined inductively as follows: The word x_i is an outer commutator word (henceforth o.c.word) of weight one. If $u = u(x_1, \ldots, x_s)$ and $v = v(x_{s+1}, \ldots, x_{s+t})$ are two o.c.words of weights s and t, respectively, then

$$w(x_1, \ldots, x_{s+t}) = [u(x_1, \ldots, x_s), v(x_{s+1}, \ldots, x_{s+t})]$$

is an o.c.word of weight $s+t$. We shall write $w = [u, v]$ and plainly for any group G we have $w(G) = [u(G), v(G)]$.

Definition 1.8

The basic commutators are defined as follows:

(i) The letters $x_1, x_2, \ldots, x_n, \ldots$ are basic commutators of weight one, ordered by setting $x_i < x_j$ if $i < j$..

(ii) If basic commutators c_i of weight $wt(c_i) < k$ are defined and ordered, then define the basic commutators of weight k by the rules:

$[c_i, c_j]$ is basic commutator of weight k, if

1. $wt(c_i) + wt(c_j) = k$.
2. $c_i > c_j$.
3. if $c_i = [c_s, c_t]$, then $c_j \geq c_t$.

Then continue and order them by setting $c > c_i$, whenever $wt(c) > wt(c_i)$ and fixing any order amongst those of weight k and finally numbering them in order.

Theorem 1.9 (P.Hall's Theorem)

Let $F = < x_1, x_2, \ldots, x_n \mid \emptyset >$ be a free group, then

$$\frac{\gamma_n(F)}{\gamma_{n+i}(F)} \quad 1 \leq i \leq n \quad .$$