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Abstract

Let G be a finite group and let $\mathrm{Aut}_p(G)$ denote the Sylow p-subgroup of $\mathrm{Aut}(G)$, the full automorphism group of G.

This thesis consists of two main parts. In the first part, we characterize those finite non-abelian p-groups G with cyclic Frattini subgroup for which $|\mathrm{Aut}_p(G)| = |G|$. In the second part, we prove a structure theorem for $\mathrm{Aut}_p(G)$, where G is a metabelian p-group of maximal class. It is shown that $\mathrm{Aut}_2(G)$ coincides $\mathrm{Aut}(G)$. For the case when $p = 3$, a structure theorem is proved for $\mathrm{Aut}(G)$ itself.

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Notations

$H \leq G$	H is a subgroup of the group G
$H \lhd G$	H is a normal subgroup of the group G
H ch K	H is a characteristic subgroup of the group G
$\|G\|$	Order of the group G
$\|G : H\|$	Index of the subgroup H in the group G
$Z(G)$	Center of the group G
$Z_i(G)$	i-th member of the upper central series of G
G'	Derived subgroup of the group G
$\Gamma_i(G)$	i-th member of the lower central series of G
$\mathcal{C}_G(H)$	Centralizer of H in G
$\mathcal{N}_G(H)$	Normalizer of H in G
$\Phi(G)$	Frattini subgroup of G
x^y	$y^{-1}xy$
$[x, y]$	$x^{-1}y^{-1}xy$
$\langle X \rangle$	Subgroup generated by X
$\mho_i(G)$	$\langle g^{p^i} \mid g \in G \rangle$ when G is a p-group
$\Omega_i(G)$	$\langle x \mid x^{p^i} = 1, x \in G \rangle$ when G is a p-group
cl(G)	Nilpotency class of G

$\mathrm{d}(G)$	Minimal number of generators of G
$\exp(G)$	Least positive integer n, such that $g^n = 1$ for each element $g \in G$
$H \rtimes K$	Semidirect product of H by K
$H * K$	Central product of H and K
$\mathrm{GL}(n, q)$	General linear group of degree n over field with q elements
$\mathrm{Hom}(G, A)$	Set of homomorphisms from G to A
$\mathrm{Aut}(G)$	Automorphism group of G
$\mathrm{Aut}_p(G)$	Sylow p-subgroup of the automorphism group of G
$\mathrm{Inn}(G)$	Inner automorphism group of G
Out(G)	$\mathrm{Aut}(G)/\mathrm{Inn}(G)$
$\mathrm{Aut}^N(G)$	Group of all automorphisms of G centralizing G/N, where $N \lhd G$
$\mathrm{Aut}^N_M(G)$	Group of automorphisms of G fixing both G/N and M elementwise
$\mathrm{Aut}^Z(G)$	Central automorphism group of G
σ_g	Inner automorphism of G induced by g
x^α	Image of x under automorphism α
$\mathbb{Z}_n$	Cyclic group of order n
$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Ring of integers
(m, n)	Greatest common divisor of integers m and n
D_{2^n}	Dihedral group of order 2^n
SD_{2^n}	Semidihedral group of order 2^n
Q_{2^n}	Generalized quaternion group of order 2^n
$C_G(H/K)$	$\{g \in G \mid [g, H] \leq K\}$
$\langle X \mid R \rangle$	Group presented by generators X and relators R
M_{2^n}	$\langle a, b \mid a^{p^{n-1}} = b^p = 1, b^{-1}ab = a^{1+p^{n-2}} \rangle$ $(n \geq 2)$

Introduction

Determining the order and the structure of the automorphism group of a finite p-group is an important problem in group theory. Perhaps the most interesting groups considered are the p-groups of maximal class. There have been a number of studies of the automorphism group of p-groups with maximal class (see for example, Baartmans and Woeppel [2], Juhász [24], Malinowska [26], Caranti and Mattarei [9], Caranti and Scoppola [10]). Some of them deal with the structure and others with the order of $\mathrm{Aut}(G)$, the automorphism group of G.

Besides the structure of the automorphism group of a p-group G, the relationship between the order of G and its automorphism group is another interesting subject. A well-known conjecture on finite p-groups states that every non-cyclic p-group G with $|G| \geq p^3$ has the property that $|G|$ divides $|\mathrm{Aut}(G)|$. This problem has been answered affirmatively in a number of special cases, and in particular for abelian groups, groups of small orders and groups of maximal class, see [13, 18, 28]. However in full generally it is still open, see [27, Problem 12.77].

This thesis consist of three chapters. In chapter 1, we give some basic results that are needed for the main results of the thesis.

In chapter 2, we characterize the finite non-abelian p-groups G with cyclic Frattini subgroup for which $|\mathrm{Aut}_p(G)| = |G|$, where $\mathrm{Aut}_p(G)$ is the Sylow p-subgroup of $\mathrm{Aut}(G)$.

Let G be a finite non-cyclic p-group of order p^n $(n \geq 3)$. It is well-known that if the Frattini subgroup $\Phi(G)$ of G is cyclic, then $|G|$ divides $|\mathrm{Aut}(G)|$ (for example see [12]). In this chapter we shall prove the following result: Let G be a finite non-abelian p-group with cyclic Frattini subgroup. Then $|G| = |\mathrm{Aut}_p(G)|$ if and only if either $G \cong SD_{16}$ or $Z(G)$ is cyclic and $|G/Z(G)| = p^2$. In particular, when $p = 2$, G has one of the following types: D_8, Q_8, SD_{16}, M_{2^n} or $L_{2^{n+2}}$ with $n > 1$ (Corollary 2.2.3 and Theorem 2.4.14). A similar description has been given by I. Malinowska in [26] for the p-groups of maximal class, in response to a problem posed by Berkovich in [5]. So our investigation solves the problem for a new class of finite non-abelian p-groups, namely the class of finite non-abelian p-groups with cyclic Frattini subgrup.

Chapter 3 is devoted to finding a structure theorem for the automorphism group of a metabelian p-group with maximal class.

Let G be a finite metabelian p-group of maximal class of order p^n and let $\Phi = \Phi(G)$ denote the Frattini subgroup of G. Juhász in [24] proved that if G is a p-group of maximal class, then $\mathrm{Aut}^{\Phi}(G)$, the group of all automorphisms of G centralizing $G/\Phi(G)$, is a split extension of $\mathrm{Inn}(G)$, the inner automorphism group of G. We here prove that when G is metabelian, a complement of $\mathrm{Inn}(G)$ in $\mathrm{Aut}^{\Phi}(G)$ is almost homocyclic of rank $p-1$; that is, it is a direct product of $p-1$ cyclic groups of order p^r or p^{r+1} for some non-negative integer r (Theorem 3.2.5). Similarly we prove a structure theorem for $\mathrm{Aut}^{P_i}(G)$ $(i \geq 2)$, where P_i is the terms of the lower central series of G and then we find the nilpotency class of $\mathrm{Aut}^{P_i}(G)$ for $i \geq 2$ (Theorem 3.2.7).

It is well-known [22, Satz III. 13.17] that the order of $\mathrm{Aut}^{\Phi}(G)$ divides p^{2n-4}. Moreover the order of $\mathrm{Aut}_p(G)$ divides p^{2n-3}. In Chapter 3 we give conditions on G for $|\mathrm{Aut}_p(G)| = p^{2n-3}$ (Corollary 3.3.5). In this case $\mathrm{Aut}_p(G)$ is a split extension of $\mathrm{Aut}^{\Phi}(G)$ by a cyclic group of order p (Theorem 3.3.8). For $p = 2$, the automorphism group is a 2-group. We give a simple proof for the structure theorem in this case (Theorem 3.4.14). It is straightforward to see that when p is odd, the (full) automorphism group $\mathrm{Aut}(G)$ of G is a split extension of $\mathrm{Aut}_p(G)$ by a subgroup of the direct product of two cyclic groups of order $p-1$, see [2, Section 1]. By using this result we prove a structure theorem for $\mathrm{Aut}(G)$ when $p = 3$ (Theorem 3.4.13).

Chapter 1

Preliminaries

In this chapter we give some basic results which will be used in the rest of the thesis.

1.1 Basic results

The following definitions are taken from [36].

Definition. Let G be a finite p-group. For each positive integer n, we define

$$\Omega_n(G) = \langle x | x \in G, x^{p^n} = 1 \rangle,$$

$$\mho_n(G) = \langle y^{p^n} | y \in G \rangle.$$

Definition. A finite p-group G is said to be *regular* if for any two elements x, y of G, there is an element c of $\mho_1(H')$ such that $x^p y^p = (xy)^p c$, where $H = \langle x, y \rangle$.

Lemma 1.1.1. [36, p. 46] *Let G be a finite p-group, where p is an odd prime. If G' is cyclic then G is regular.*

Theorem 1.1.2. [36, p. 47] *Let G be a finite regular p-group. Then the following properties hold.*

(i) *The subgroup* $\Omega_n(G)$ *is the set of elements of order at most* p^n.

(ii) *The subgroup* $\mho_n(G)$ *is the set of* p^n*-th power of the elements of* G.

(iii) $|G:\Omega_n(G)| = |\mho_n(G)|$.

Lemma 1.1.3. *Let* G *be a finite* p*-group. Then*

(i) $\Phi(G) = G'\mho_1(G)$,

(ii) *if* $p = 2$ *then* $\Phi(G) = \mho_1(G)$,

(iii) $G/\Phi(G)$ *is elementary abelian,*

(iv) *if* $G = H \times K$ *then* $\Phi(G) = \Phi(H) \times \Phi(K)$.

Theorem 1.1.4. [19, Theorem 4.4, p.193] *Let* G *be a non-abelian* p*-group of order* p^n *with cyclic subgroup of index* p*. Then* G *is isomorphic to one of the following groups:*

(i) $M_{p^n} = \langle a, b | a^{p^{n-1}} = b^p = 1, b^{-1}ab = a^{1+p^{n-2}} \rangle$, *where* $n \geq 2$ *and* $n > 3$ *if* $p = 2$*; moreover* $|G'| = p$, $Z(G) = \Phi(G) \cong \mathbb{Z}_{p^{n-2}}$ *and* $\mathrm{cl}(G) = 2$*;*

(ii) $D_{2^n} = \langle a, b | a^{2^{n-1}} = b^2 = 1, (ab)^2 = 1 \rangle$, *the dihedral group of order* 2^n*;* $Q_{2^n} = \langle a, b | a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1} \rangle$, *the generalized quaternion group of order* 2^n*, both with* $n \geq 3$*;* $SD_{2^n} = \langle a, b | a^{2^{n-1}} = b^2 = 1, bab = a^{-1+2^{n-2}} \rangle$, $n \geq 4$ *the semidihedral group; moreover* $G' = \Phi(G) = \langle a^2 \rangle \cong \mathbb{Z}_{2^{n-2}}$, $Z(G) \cong \mathbb{Z}_2$ *and* $\mathrm{cl}(G) = n - 1$.

The following lemma is easily proved by considering the presentation of the corresponding groups given in Theorem 1.1.4.

Lemma 1.1.5. *If G is either D_{2^n} $(n \geq 3)$ or Q_{2^n} $(n > 3)$, then $|\mathrm{Aut}(G)| = 2^{2n-3}$ and $|\mathrm{Aut}(Q_8)| = 24$. Furthermore if $G = SD_{2^n}$ with $n \geq 4$, then $|\mathrm{Aut}(G)| = 2^{2n-4}$.*

We have the following definition taken from [25].

Definition. The *upper central series* of G is the series

$$1 = Z_0(G) < Z_1(G) < Z_2(G) < \dots$$

of subgroups of G defined inductively by $Z_i(G)/Z_{i-1}(G) = Z(G/Z_{i-1}(G))$ for $i > 0$. The *lower central series* of G is the descending series

$$G = \Gamma_1(G) > \Gamma_2(G) > \dots$$

of subgroups of G, where $\Gamma_{i+1}(G) = [\Gamma_i(G), G]$ for $i > 1$.

Theorem 1.1.6. [22, Satz III. 2.13] *Let G be a finite group. Then*

(i) $\exp(\Gamma_{i+1}(G)/\Gamma_{i+2}(G))$ *divides* $\exp(\Gamma_i(G)/\Gamma_{i+1}(G))$ *for* $i \geq 1$,

(ii) $\exp(Z_{i+1}(G)/Z_i(G))$ *divides* $\exp(Z_i(G)/Z_{i-1}(G))$ *for* $i \geq 1$.

Definition. A finite p-group G is called *extra-especial* if $G' = Z(G) = \Phi(G) \cong \mathbb{Z}_p$.

Obviously every non-abelian group of order p^3 is extra-especial.

The following lemma indicates the basic properties of $|\mathrm{Hom}(A, B)|$, where A is an arbitrary finite group and B is a finite abelian p-group.

Lemma 1.1.7. *Let G be a group and let A, B, C be abelian groups. Then*

(i) $|\mathrm{Hom}(A, B)| = |\mathrm{Hom}(B, A)|$,

(ii) $|\mathrm{Hom}(A, B \times C)| = |\mathrm{Hom}(A, B) \times \mathrm{Hom}(A, C)|$,

(iii) $|\mathrm{Hom}(G, A)| = |\mathrm{Hom}(G/G', A)|$.

Lemma 1.1.8. *Let G be a group and let H, K be subgroups of G. If $G = H \cup K$ then $H \leq K$ or $K \leq H$.*

Definition. Let G be a group and $N \lhd G$. We define $\mathrm{Aut}^N(G)$ to be the set of all automorphisms α of G, such that $g^{-1}g^{\alpha} \in N$ for all g in G.

In fact, $\mathrm{Aut}^N(G)$ consists of those automorphisms which induce the identity automorphism on G/N. It is easily seen that $\mathrm{Aut}^N(G) \trianglelefteq \mathrm{Aut}(G)$ when N is a characteristic subgroup of G.

Theorem 1.1.9. [22, Satz III. 3.17] *Let G be a p-group of order p^n. If $|G/\Phi(G)| = p^r$, then $|\mathrm{Aut}(G)|$ divides $p^{r(n-r)}|\mathrm{GL}(r,p)|$ and $|\mathrm{Aut}^{\Phi}(G)|$ divides $p^{r(n-r)}$.*

Lemma 1.1.10. *Let G be a p-group. Then*

(i) $\mathrm{Inn}(G) \leq \mathrm{Aut}^{\Phi}(G)$,

(ii) *if H ch G, then* $\mathrm{Aut}(G)/\mathrm{Aut}^H(G) \hookrightarrow \mathrm{Aut}(G/H)$,

(iii) *if H, K ch G and $H \leq K$, then* $\mathrm{Aut}^K(G)/\mathrm{Aut}^H(G) \hookrightarrow \mathrm{Aut}^{K/H}(G/H)$.

Lemma 1.1.11. *Let H and K be normal subgroups of a group G and $H = \langle X \rangle$, $K = \langle Y \rangle$. Then $[H, K] = \langle [x, y]^g | g \in G, x \in X, y \in Y \rangle$.*

Theorem 1.1.12. [23, Theorem p.44] *Suppose that we are given a presentation $\langle X|R \rangle$ for a finite group G, and a mapping $\theta : X \to G$. Then θ extends to an endomorphism* *of G if and only if for all $x \in X$ and all $r \in R$ the result of substituting x^θ for x in r yields the identity of G. Furthermore if, in addition X^θ generates G then θ extends to an automorphism of G.*

1.2 Central automorphisms

In this section we introduce the group of central automorphisms of a finite group and give some basic properties of this group.

Definition. Let G be a group. An automorphism σ is said to be *central* if $g^{-1}g^\sigma \in Z(G)$ for all $g \in G$. The set of all central automorphisms of G denoted by $\mathrm{Aut}^Z(G)$, form a normal subgroup of $\mathrm{Aut}(G)$.

Lemma 1.2.1. *Let G be a group. Then*

(i) *every central automorphism fixes G' elementwise,*

(ii) $\mathrm{Aut}^Z(G)$ *is the centralizer of* $\mathrm{Inn}(G)$ *in* $\mathrm{Aut}(G)$,

(iii) $\mathrm{Aut}^Z(G) \cap \mathrm{Inn}(G) = Z(\mathrm{Inn}(G))$.

Definition. A non-abelian group that has no non-trivial abelian direct factor is said to be *purely non-abelian.*

Theorem 1.2.2. [1, Theorem 1] *Let G be a finite purely non-abelian group. Then there is a 1-1 correspondence between* $\mathrm{Hom}(G, Z(G))$ *and* $\mathrm{Aut}^Z(G)$. *Therefore we have* $|\mathrm{Hom}(G/G', Z(G))| = |\mathrm{Aut}^Z(G)|$.

For finite groups having a non-trivial abelian direct factor, we have the following result.

Lemma 1.2.3. [12, Lemma 1] *Let $G = H \times K$, where H is abelian and K is purely non-abelian. Then* $|\mathrm{Aut}^Z(G)| = |\mathrm{Aut}(H)||\mathrm{Aut}^Z(K)||\mathrm{Hom}(K,H)||\mathrm{Hom}(H,Z(K))|$.

Lemma 1.2.4. *Let G be one of the groups D_8, Q_8 or M_{2^n}. Then G has a non-central automorphism of order 2 fixing G' elementwise.*

Proof. By considering the presentation of G given in Theorem 1.1.4, we may see that if G is either D_8 or Q_8, then the automorphism α defined by $a^\alpha = a^{-1}$, $b^\alpha = ab$ is of order 2 fixing G' elementwise. Also when $G = M_{2^n}$, the automorphism β defined by $a^\beta = ba$ and $b^\beta = b$ is the desired automorphism. □

1.3 p-groups of class 2

Lemma 1.3.1. *Let G be a nilpotent group of class 2. Then for all x, y, $w \in G$, $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ we have:*

(i) $[x, yw] = [x,y][x,w]$,

(ii) $[xy, w] = [x,w][y,w]$,

(iii) $[x, y^m] = [x^m, y] = [x,y]^m$,

(iv) $(xy)^n = x^n y^n [y,x]^{n(n-1)/2}$.

Lemma 1.3.2. *Let G be a finite nilpotent group of class 2. Then $\exp(G') = \exp(G/Z)$ and in the decomposition of $G/Z(G)$ into direct products of cyclic groups at least two factors of maximal order must occur.*

Proof. Suppose that $G/Z(G)$ is generated by $\{Zx_1, Zx_2, \ldots, Zx_n\}$. Therefore G' is generated by $\{[x_i, x_j], 1 \leq i, j \leq n\}$. We then apply Lemma 1.3.1. □

We have the following lemma which is a special case of [1, Lemma 1].

Lemma 1.3.3. *Let G be a purely non-abelian p-group of class 2, where p is an odd prime. Suppose that $Z(G)$ is cyclic and $G' = \langle u \rangle \cong \mathbb{Z}_p$. Then*

(i) *$u = [g, h]$, where $g, h \in G$ and $|h| = p$,*

(ii) *on setting $H = \langle g, h \rangle$, we have $G = H\mathcal{C}_G(H)$,*

(iii) *the map α defined by $g^\alpha = gh$, $h^\alpha = h$ and $x^\alpha = x$ for $x \in \mathcal{C}_G(H)$ is an automorphism of G of order p.*

Proof. (i) There exist $g, h_1 \in G$ such that $[g, h_1] \neq 1$, so we may write $u = [g, h_1]$ since $|G'| = p$. Suppose that $Z(G) = \langle z \rangle$ and z has order p^m. By Lemma 1.3.2, $g^p = z^s$ and $h_1^p = z^t$, where t, s are non-negative integers. We may assume that $t \equiv rs \pmod{p^m}$. Now on setting $h = g^{-r}h_1$, we have $[g, h] = u$ and $h^p = 1$.

(ii) For any $x \in G$ we have $[g, x] = u^a$ and $[h, x] = u^b$, where $0 \leq a, b < p$. Then $h^{-a}g^bx \in \mathcal{C}_G(H)$. Hence $x = g^{-b}h^a(h^{-a}g^bx) \in H\mathcal{C}_G(H)$, completing the proof.

(iii) To prove that α is an automorphism, we need to show $(hg)^\alpha = h^\alpha g^\alpha$, $(xg)^\alpha = x^\alpha g^\alpha$ for $x \in \mathcal{C}_G(H)$ and $g^p = (g^i)^\alpha (g^{p-i})^\alpha$ for $0 \leq i < p$. We can check these equations by direct computation, bearing in mind that $(xy)^p = x^py^p$ for all $x, y \in G$. □

1.4 Central product

In this section we give some basic results for the central product of two subgroups of a given group.

Definition. A group G is said to be a *central product* of two subgroups H and K, if H and K commute elementwise and $G = HK$. In this situation we write $G = H * K$.

The central product of more than two subgroups can be defined similarly. Suppose that $H_1, \dots, H_n$ are subgroups of a group G, which satisfy the following conditions: if $i \neq j$ then H_i and H_j commute elementwise and $G = H_1 \dots H_n$. In this case, we call G a *central product* of the subgroups $H_1, \dots, H_n$.

Theorem 1.4.1. [35, p.137] *Let G be a group which is a central product of two subgroups H and K. Set $D = H \cap K$.*

(i) *Both H and K are normal subgroups of G.*

(ii) *$D \leq Z(H)$ and $D \leq Z(K)$.*

(iii) *Let H_1 be a group which is isomorphic to H via an isomorphism $\alpha : H \to H_1$. Similarly, let $\beta : K \to K_1$ be an isomorphism. On setting $G_1 = H_1 \times K_1$, the function ψ defined by $(h_1, k_1)^\psi = h_1^{\alpha^{-1}} k_1^{\beta^{-1}}$ is a homomorphism from G_1 onto G with $\mathrm{Ker}(\psi) = \{(x^\alpha, x^{-\beta}) | x \in D\}$, $D \cong \mathrm{Ker}(\psi) \leq Z(G_1)$ and $G_1/\mathrm{Ker}(\psi) \cong G$.*

Lemma 1.4.2. *Let G be a finite p-group and H, K be subgroups of G such that $G = H * K$. Then*

(i) $G' = H'K'$,

(ii) $Z(G) = Z(H)Z(K)$,

(iii) $\Phi(H)\Phi(K) \leq \Phi(G)$,

(iv) *if* $\mathrm{cl}(H) = \mathrm{cl}(K) = 2$, *then* $\mathrm{cl}(G) = 2$.

Lemma 1.4.3. *Suppose that H, K and L are subgroups of a group G. Then*

(i) $(H * K) * L = H * (K * L)$,

(ii) $(H * K) \times L = H * (K \times L)$.

Theorem 1.4.4. [22, Satz III. 13.7] *Suppose that G is a non-abelian p-group, $G/Z(G)$ is elementary abelian and $Z(G)$ is cyclic. Then*

(i) $|G'| = p$,

(ii) *G is a central product of its non-abelian subgroups $G_1, \ldots, G_k$ such that $Z(G) = Z(G_i)$ and $|G_i/Z(G_i)| = p^2$ for $1 \leq i \leq k$.*

Lemma 1.4.5. *Let G be a finite group and let A, B be subgroups of G such that G is a central product of A and B. If α is a non-central automorphism of A whose restriction to the intersection of A and B is the identity, then α can be extended to a non-central automorphism of G.*

Proof. We define the map α^* from G into G by $(ab)^{\alpha^*} = a^\alpha b$ for all $a \in A$ and $b \in B$. Then α^* is the desired automorphism which is an extension of α. □

Theorem 1.4.6. [7, Theorem 3.2] *Let G be a finite p-group which is a central product of non-trivial subgroups N and H with $N < G$, $H \leq G$ and assume that $N \cap [H, H] = 1$. Then $|\mathrm{Aut}_p(G)| \geq p|\mathrm{Aut}_p(N)||\mathrm{Aut}_p(G/N)|$.*