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ALMOST PERFECT AND GENERALIZED PERFECT RINGS

By

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To my mother and father

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ABSTRACT

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Bican et al. in [9] have proved that every module over an arbitrary ring has a flat cover. In this thesis we shall study rings over which flat covers of finitely generated modules are projective. We call a ring R right almost perfect if every flat right R-module is projective relative to R. It turns out that a ring R is right almost perfect if and only if it is semiperfect and flat covers of finitely generated right R-modules are finitely generated, equivalently, flat covers of finitely generated right R-modules are projective. We shall show that the class of almost perfect rings is properly between the classes of perfect and semiperfect rings. We also outline some new characterizations of perfect rings. For example, we show that a ring R is right perfect if and only if every finitely cogenerated right R-module has a projective cover.

Also we call a ring R right generalized perfect if every right R-module is a superfluous epimorphic image of a flat right R-module. We shall investigate some properties of these rings and show that a commutative generalized perfect ring is a max ring. Finally, we find some classes of modules which are superfluous epimorphic images of flat modules.

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INTRODUCTION

Since Eckmann and Schopf in [14] proved the existence of injective envelopes for modules over any associative ring and Matlis in [34] gave the structure theorem of injective modules over Noetherian rings, the notions of injective modules and injective envelopes (injective hulls) have played an important role in the theory of modules and rings and have had a great impact on homological algebra and commutative algebra. In an attempt to dualize the concept of injective envelopes, Bass in [7] successfully studied projective covers of modules and initiated the study of perfect rings. These rings possess nice theoretical and homological properties. The harmony between the global characterizations and the internal descriptions of these rings exhibits the beauty and the nature of structures in algebra.

Motivated by injective envelopes and projective covers, several other notions of envelopes and covers have been defined and investigated. For instance, Warfield in [39] studied the pure injective envelopes of modules and Enochs in [17] defined the torsion free coverings of modules and proved their existence over any integral domain. Considering various kinds of envelopes and covers, there is a natural question: How can we define envelopes or covers in a general setting? Enochs in [18] first noticed the categorical version of injective envelopes and made a general definition of envelopes and covers by diagrams for a given class of modules.

Let Ω be a class of R-modules which is closed under isomorphic copies. A homomorphism $f : F \longrightarrow M$ with $F \in \Omega$ is called an Ω-precover of the R-module M if for each homomorphism $g : G \longrightarrow M$ with $G \in \Omega$, there exists $h : G \longrightarrow F$ such that $fh = g$. An Ω-precover $f : F \longrightarrow M$ is said to be an Ω-cover if every endomorphism l of F with $fl = f$ is an automorphism of F. An Ω-(pre)envelope of a module is defined dually. Now if Ω is the class of all flat modules, an Ω-cover is usually called a flat cover. Enochs in [18] conjectured that every module over an associative ring admits a flat cover, because many properties of flat modules are highly dualized counterparts of those for injective modules. To prove this conjecture, several authors extensively studied flat covers and related notions and solved the conjecture in some special cases, see, for example, [8, 10, 16, 18, 19, 42, 43]. Finally after two decades, Bican et al. in [9] proved the existence of flat covers over any associative ring in two different ways.

This thesis is mainly devoted to study two classes of rings. First, rings over which flat covers of finitely generated modules are projective; second, rings over which every module is a superfluous epimorphic image of a flat module.

In Chapter 1 we give some general terminologies and preliminary results which are needed in other chapters. In Chapter 2 we briefly present basic properties of perfect and semiperfect rings. Concerning the dual concept of the statement that "if every finitely generated R-module has a projective cover, then R is semiperfect", we prove that "a ring R is right perfect in case each finitely cogenerated right R-module has a projective cover". Also we study quasi-perfect rings (which introduced by Camillo and Xue in [11]) and we shall see that they form a class of rings strictly between the class of perfect and semiperfect rings.

We know that if a finitely generated R-module M has a projective cover $\varphi : P \longrightarrow M$, then P is finitely generated (see [4, Lemma 17.17]), however, we show that for the flat cover this is not the case. In Chapter 3 we introduce a class of rings (i.e., right almost perfect rings) for which the above holds. An R-module M is said to be R-injective in case any homomorphism $f : I \longrightarrow M$, where I is a right ideal of R, can be extended to $\bar{f} : R \longrightarrow M$. By Baer's Theorem, any R-injective module is injective (see, for example, [4, Lemma 18.3]). Dually R-projective modules are defined, but R-projective modules need not be projective. We call a ring R right almost perfect in case all flat right R-modules are R-projective. Note that a ring R is right perfect if and only if all flat right R-modules are projective. In particular, right perfect rings are right almost perfect. It turns out that a ring R is right almost perfect if and only if R is semiperfect and flat covers of finitely generated right R-modules are finitely generated. Also we show that almost perfectness is not left-right symmetric and that the class of almost perfect rings is strictly between the class of perfect and semiperfect rings, meanwhile, it is different from the class of quasi-perfect rings.

Bass in [7] called a ring R right perfect if every right R-module is a superfluous epimorphic image of a projective module. However, we shall replace the word "projective" by "flat" in Bass' definition and get some new related results. In Chapter 4 we say that a module has a G-flat cover, if it is a superfluous epimorphic image of a flat module. Moreover, we call a ring R right generalized perfect if every right R-module has a G-flat cover and we investigate some properties of these rings. For example, we show that the Jacobson radical of a right generalized perfect ring is right T-nilpotent. Also it turns out that a commutative generalized perfect ring is a max ring. Finally, in Chapter 5 we define minimal G-flat cover of a module and show that if a module has a G-flat cover, then it has a minimal G-flat cover. Finally we find some classes of modules which have G-flat covers. For instance, finitely cogenerated modules and cyclic modules over commutative max rings have G-flat covers.

Chapter 1

PRELIMINARIES

1 Preliminaries

Throughout this thesis, all rings are associative with identity and all modules are unitary right modules unless stated otherwise. Homomorphisms between modules are written and composed on the opposite side of scalars. Semisimple rings are in the sense of Wedderburn and Artin: these are rings which are semisimple as right (or left) modules over themselves. By a regular ring, we mean a ring R such that $x \in xRx$ for any $x \in R$ (i.e., a von Neumann regular ring). For a ring R, let Mod-R denote the category of all right R-modules and $J(R)$ be the Jacobson radical of R. For a module M_R, the notation $K \leq M$ means that K is a submodule of M and $K \ll M$ means that K is a superfluous submodule of M in the sense that $K + L \neq M$ for any proper submodule L of M. Also $K \trianglelefteq M$ means that K is an essential submodule of M, that is, $K \cap L \neq 0$ for any nonzero submodule L of M. We write $\mathrm{soc}(M_R)$, $\mathrm{rad}(M_R)$ and $E(M_R)$ for the socle, the Jacobson radical and the injective hull of M_R, respectively. Note that

$$\mathrm{rad}(M_R) = \bigcap\{K : K \text{ is a maximal submodule of } M\} = \sum\{L : L \ll M\}$$

and

$$\mathrm{soc}(M_R) = \sum\{K : K \text{ is a minimal submodule of } M\} = \bigcap\{L : L \trianglelefteq M\}.$$

For the proof, see [4, Propositions 9.7 and 9.13]. A monomorphism $f : M \longrightarrow N$ between modules M and N is said to be an essential monomorphism if $\mathrm{im}(f) \trianglelefteq N$. Dually, an epimorphism $f : M \longrightarrow N$ is said to be a superfluous epimorphism if $\ker(f) \ll M$.

Enochs in [18] gave the following general definition of covers and envelopes.

Definition 1.1 Let R be a ring and Ω be a class of R-modules which is closed under isomorphic copies. An Ω-precover of an R-module M is a homomorphism $\varphi : F \longrightarrow M$ with $F \in \Omega$ such that for any homomorphism $\psi : G \longrightarrow M$ with $G \in \Omega$, there exists $\mu : G \longrightarrow F$ such that $\varphi\mu = \psi$. An Ω-precover $\varphi : F \longrightarrow M$ is said to be an Ω-cover if every endomorphism λ of F with $\varphi\lambda = \varphi$ is an automorphism of F. Dually, Ω-preenvelope and Ω-envelope of an R-module are defined.

If $\varphi : F \longrightarrow M$ is an Ω-(pre)cover of M, we usually refer to F as an Ω-(pre)cover of M. Now we state some elementary properties of covers which are needed in the sequel. Note that similar results also hold for envelopes.

Proposition 1.2 *Let $F_1, F_2 \in \Omega$.*

(a) If $f_1 : F_1 \longrightarrow M$ and $f_2 : F_2 \longrightarrow M$ are two different Ω-covers of M, then there is an isomorphism $g : F_1 \longrightarrow F_2$ such that $f_2 g = f_1$.

(b) Let $f_1 : F_1 \longrightarrow M$ be an Ω-cover of M and $f_2 : F_2 \longrightarrow M$ be a homomorphism. If there is an isomorphism $g : F_1 \longrightarrow F_2$ such that $f_2 g = f_1$, then $f_2 : F_2 \longrightarrow M$ is an Ω-cover of M.

Proof. (a) Since both F_1 and F_2 are Ω-covers of M, there are homomorphisms $g : F_1 \longrightarrow F_2$ and $h : F_2 \longrightarrow F_1$ such that $f_1 = f_2 g$ and $f_2 = f_1 h$. Thus $f_1 = f_1 hg$ and $f_2 = f_2 gh$. By the definition of the Ω-cover, gh and hg are automorphisms of F_2 and F_1, respectively. Therefore g and h are isomorphisms.

(b) First we show that $f_2 : F_2 \longrightarrow M$ is an Ω-precover of M. Let $H \in \Omega$ and $h : H \longrightarrow M$ be a homomorphism. Since $f_1 : F_1 \longrightarrow M$ is an Ω-precover of M, there is $k : H \longrightarrow F_1$ such that $h = f_1 k = f_2 g k$. Now suppose that $l : F_2 \longrightarrow F_2$ with $f_2 l = f_2$. Thus $f_1 g^{-1} l = f_1 g^{-1}$ and so $f_1 g^{-1} l g = f_1$. As $f_1 : F_1 \longrightarrow M$ is an Ω-cover of M, $g^{-1} l g$ is an automorphism of F_1 and hence l is an automorphism of F_2. Therefore. $f_2 : F_2 \longrightarrow M$ is an Ω-cover of M. □

Suppose that $f : F \longrightarrow M$ is an Ω-precover of M and $M = M_1 \oplus M_2$. Let $p : M \longrightarrow M_1$ and $q : M_1 \longrightarrow M$ be the natural projection and injection, respectively. Then $pf : F \longrightarrow M_1$ is an Ω-precover of M_1. For, if $g : G \longrightarrow M_1$ is a homomorphism with $G \in \Omega$, then $qg : G \longrightarrow M$. Since $f : F \longrightarrow M$ is an Ω-precover of M, there is $h : G \longrightarrow F$ such that $fh = qg$. Therefore, $pfh = pqg = g$.

Proposition 1.3 *Suppose that M has an Ω-cover and $g : G \longrightarrow M$ is an Ω-precover of M. Then $G = G_1 \oplus G_2$ where $G_2 \subseteq \ker(g)$ and $g|_{G_1} : G_1 \longrightarrow M$ is an Ω-cover of M.*

Proof. Let $f : F \longrightarrow M$ be an Ω-cover of M. There are homomorphisms $h : F \longrightarrow G$ and $k : G \longrightarrow F$ such that $gh = f$ and $fk = g$. So $f = fkh$. Since $f : F \longrightarrow M$ is an Ω-cover of M, it follows that kh is an automorphism of F and so $G = \mathrm{im}(h) \oplus \ker(k)$. Put $G_1 = \mathrm{im}(h)$ and $G_2 = \ker(k)$. As $fk = g$, we have $G_2 \subseteq \ker(g)$ and $h : F \longrightarrow G_1$ is an isomorphism with $(g|_{G_1})h = f$. Therefore, by Proposition 1.2, $g|_{G_1} : G_1 \longrightarrow M$ is an Ω-cover of M. □

The proof of the following result is an immediate consequence of the properties of direct product.

Theorem 1.4 *Let $\varphi_i : F_i \longrightarrow M_i$ be an Ω-precover of M_i for any $i \in I$. If $\prod_{i \in I} F_i \in \Omega$, then $\prod \varphi_i : \prod F_i \longrightarrow \prod M_i$ is an Ω-precover of $\prod_{i \in I} M_i$.*

Note that in Theorem 1.4, even if each $\varphi_i : F_i \longrightarrow M_i$ is an Ω-cover, then $\prod_{i \in I} F_i$ may fail to be an Ω-cover of $\prod_{i \in I} M_i$. For a counterexample see [43, Theorem 1.3.9]. However, if I is a finite set, we have the following result ([43, Theorem 1.2.10]).

Theorem 1.5 *If $\varphi_i : F_i \longrightarrow M_i$ is an Ω-cover of M_i for $1 \leq i \leq n$ and Ω is closed under finite direct sums, then $\bigoplus \varphi_i : \bigoplus F_i \longrightarrow \bigoplus M_i$ is an Ω-cover of $\bigoplus_{i=1}^{n} M_i$.*

A projective cover of an R-module M is an epimorphism $\varphi : P \longrightarrow M$ where P is a projective R-module and $\ker(\varphi) \ll P$ (i.e., φ is a superfluous epimorphism). Now let Ω be the class of all projective R-modules. Then we have the consistency between the notion of projective cover and the notion of Ω-cover for a module M.

Theorem 1.6 *Let Ω be the class of all projective right R-modules and let $\varphi : P \longrightarrow M$ be a homomorphism with $P \in \Omega$. Then the following statements are equivalent:*

(a) $\varphi : P \longrightarrow M$ is a projective cover of M;

(b) $\varphi : P \longrightarrow M$ is an Ω-cover of M.

Proof. (a) $\Longrightarrow$ (b). Let $Q \in \Omega$ and $\alpha : Q \longrightarrow M$ be a homomorphism. Since φ is an epimorphism, by the projectivity of Q, there is $\gamma : Q \longrightarrow P$ with $\varphi\gamma = \alpha$. Thus $\varphi : P \longrightarrow M$ is an Ω-precover. Now suppose that λ is an endomorphism of P with $\varphi\lambda = \varphi$. Thus $\mathrm{im}(\lambda) + \ker(\varphi) = P$. Since $\ker(\varphi) \ll P$, λ is an epimorphism. By the projectivity of P, there exists $\mu : P \longrightarrow P$ with $\lambda\mu = \mathrm{id}_P$ (the identity endomorphism of P). Therefore, $P = \mathrm{im}(\mu) \oplus \ker(\lambda)$. But $\varphi\lambda = \varphi$ implies that $\ker(\lambda) \subseteq \ker(\varphi) \ll P$. Hence $\ker(\lambda) = 0$ and so λ is an automorphism of P. Consequently, $\varphi : P \longrightarrow M$ is an Ω-cover of M.

(b) $\Longrightarrow$ (a). There exists an epimorphism $\psi : F \longrightarrow M$ for some projective module F. Since $\varphi : P \longrightarrow M$ is an Ω-(pre)cover of M, $\psi = \varphi\lambda$ for some homomorphism $\lambda : F \longrightarrow P$. Now the surjectivity of ψ implies that φ is an epimorphism. To show that $\ker(\varphi) \ll P$, suppose that $\ker(\varphi)+L = P$ for some $L \leq P$. Thus $\varphi|_L : L \longrightarrow M$ is an epimorphism. As P is a projective module, there is $\mu : P \longrightarrow L \subseteq P$ such that $(\varphi|_L)\mu = \varphi$. Therefore, $\varphi\mu = \varphi$. Now by the definition of an Ω-cover, μ is an automorphism of P and so $P = \mathrm{im}(\mu) \subseteq L$. Consequently, $\varphi : P \longrightarrow M$ is a superfluous epimorphism and so is a projective cover of M. $\square$

Remark 1.7 Let Ω be the class of all injective R-modules and $\varphi : M \longrightarrow E$ be a homomorphism with $E \in \Omega$. Then as in Theorem 1.6 we can show that $\varphi : M \longrightarrow E$ is an Ω-envelope of M if and only if it is an injective envelope of M (i.e., φ is an essential monomorphism).

Eckmann and Schopf [14] proved that over any ring every module M has an injective envelope denoted by $E(M)$. But for projective cover this is not the case. Bass in [7] called a ring R right perfect in case every right R-module has a projective cover. He also gave some internal and homological characterizations for these rings.

Now let Ω be the class of all flat R-modules. From now on, we call an Ω-cover (or Ω-precover) of an R-module M, a flat cover (or flat precover) of M. Enochs in [18] conjectured that over an arbitrary ring, all modules have flat covers. Several authors worked on this conjecture, see, for example, [8, 10, 16, 18, 19, 42, 43]. Finally it was solved by Bican et al. in 2001 in two completely different ways (see [9]).

Theorem 1.8 *Let R be a ring. Then any R-module has a flat cover.*

Proof. See [9, Theorem 3]. □

The following result is true for any class Ω, which is closed under extensions. However, we prove it for the class of flat modules which also has this property (Proposition 1.16).

Lemma 1.9 *Let $\varphi : F \longrightarrow M$ be a flat cover of the R-module M and $K = \ker(\varphi)$. Then for any flat module G, $\mathrm{Ext}^1_R(G, K) = 0$ (i.e., any exact sequence of R-modules of the form $0 \longrightarrow K \longrightarrow L \longrightarrow G \longrightarrow 0$ splits).*

Proof. Let G be a flat R-module. Consider an exact sequence of R-modules of the form $0 \longrightarrow K \xrightarrow{v} L \xrightarrow{p} G \longrightarrow 0$. Let $\alpha : K \longrightarrow F$ be the inclusion map. Then we have the pushout diagram of the monomorphisms $v : K \longrightarrow L$ and $\alpha : K \longrightarrow F$.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \xrightarrow{\alpha} & F & \xrightarrow{\varphi} & M & \longrightarrow & 0 \\
 & & \downarrow v & & \downarrow f & & \| & & \\
0 & \longrightarrow & L & \xrightarrow{h} & P & \xrightarrow{\sigma} & M & \longrightarrow & 0 \\
 & & \downarrow p & & \downarrow q & & & & \\
 & & G & = & G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Note that G and F are flat R-modules and hence P is too. Since $\varphi : F \longrightarrow M$ is a flat cover, there is $g : P \longrightarrow F$ such that $\varphi g = \sigma$. Hence $\varphi g f = \sigma f = \varphi$. Therefore, gf must be an automorphism of F and so $\varphi = \varphi(gf)^{-1}$. Thus $\varphi(gf)^{-1}gh = \varphi gh = \sigma h = 0$. Therefore, $\mathrm{im}\,((gf)^{-1}gh) \subseteq \ker(\varphi) = \mathrm{im}(\alpha)$. So we can define $u = \alpha^{-1}(gf)^{-1}gh : L \longrightarrow K$. Therefore, $uv = \alpha^{-1}(gf)^{-1}ghv =$

$\alpha^{-1}(gf)^{-1}gf\alpha = \alpha^{-1}\alpha = \mathrm{id}_K$. This implies that $0 \longrightarrow K \longrightarrow L \longrightarrow G \longrightarrow 0$ splits. □

Definition 1.10 An R-module C is said to be cotorsion if for any flat R-module F, $\mathrm{Ext}^1_R(F, C) = 0$.

By Lemma 1.9, we see that if $\varphi : F \longrightarrow M$ is a flat cover of an R-module M, then $\ker(\varphi)$ is a cotorsion module. However, for the converse we have the following result.

Lemma 1.11 *Let F be a flat R-module and $\varphi : F \longrightarrow M$ be an epimorphism. If $\ker(\varphi)$ is cotorsion, then $\varphi : F \longrightarrow M$ is a flat precover of M, which is called a special flat precover.*

Proof. Let G be a flat R-module and $K = \ker(\varphi)$. Then the exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ induces the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(G, K) \longrightarrow \mathrm{Hom}_R(G, F) \longrightarrow \mathrm{Hom}_R(G, M) \longrightarrow \mathrm{Ext}^1_R(G, K) = 0.$$

Therefore, $\mathrm{Hom}_R(G, F) \longrightarrow \mathrm{Hom}_R(G, M)$ is an epimorphism (i.e., for any homomorphism $\alpha : G \longrightarrow M$, there is a homomorphism $\lambda : G \longrightarrow F$ such that $\varphi\lambda = \alpha$) and so $\varphi : F \longrightarrow M$ is a flat precover of M. □

By the elementary properties of the Ext functor, see [37], it is clear that the class of cotorsion modules is closed under extensions, finite direct sums, arbitrary direct products and direct summands.

A short exact sequence $0 \longrightarrow A \xrightarrow{\varphi} B \longrightarrow C \longrightarrow 0$ of right R-modules is said to be pure (exact) if

$$0 \longrightarrow A \otimes_R M \longrightarrow B \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0$$