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CUT-POINT SPACES

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Abstract

A cut-point space is defined to be a topological space that is connected but the removal of any one of its points leaves it disconnected. In this thesis, we study the cut-point spaces. In chapter 2, it is shown that every cut-point space has an infinite number of closed points. Also, it is proved that every cut-point space is non-compact. Moreover, a characterization of the Khalimsky line with respect to cut-point spaces is given. In chapter 3, topological constructions are used to obtain new cut-point spaces from the old ones. In chapter 4, cut-point spaces with special (topological or algebraic) structures are studied. In chapter 5, the covering dimension of cut-point spaces is studied briefly.

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Chapter 1

Introduction

The real line **R** is a source of intuition in topology. Many other familiar topological spaces can be obtained from **R** by topological constructions. It has the following properties:

(*a*) it is connected but the removal of any one of its points leaves it disconnected;

(*b*) it is metrizable;

(*c*) its topology can be generated by a linear ordering.

Conversely, it can be proved that every topological space with the above properties is homeomorphic to **R** (see Theorem 4.9). Conditions (*b*) and (*c*) are too strong. They impose structures on the topological space, so this characterization of **R** seems somehow extrinsic. On the other hand, there is an increasing interest in studying general topological spaces without special structures or restricting conditions (such as seperation axioms). A well-known example of such spaces is the space of prime ideals of a ring with the Zariski topology on it [1]. Two more recent examples are the Scott topology on the complete partially ordered sets which appears in denotational semantics [3] and the Khalimsky topology on digital k-space (the product $\mathbf{Z}^k$ of k copies of the Khalimsky line **Z**) which is useful in image processing [6,7].

In this dissertation we study the topological spaces that satisfy condition (*a*), and call them cut-point spaces. In chapter 2, a cut-point space is defined again formally and some examples are given. It is shown that every cut-point space has an infinite number of closed points. Also, it is proved that every cut-point space is non-compact. To prove the latter, we need the most general form of the non-cut point existence theorem. The special case of this theorem for metric topological spaces is proved in [8]. A proof of the theorem for T_1 topological spaces can be found in [5] (see also [9]). An irreducible cut-point space is defined naturally as a cut-point space whose proper subsets are not cut-point spaces. It is shown that an irreducible cut-point space is necessarily homeomorphic to the Khalimsky line (see Example 2.5 for the definition of the Khalimsky line). This result may also be viewed as a straightforward characterization of the Khalimsky line. Objects in n-dimensional digital images have sometimes been regarded as subspaces of the product of n copies of the Khalimsky line [6,7]. In chapter 3, we consider the subspaces, quotients, products, and continuous images of cut-point spaces to obtain new cut-point spaces from the old ones. We will see that there are many difficulties in this route. In chapter 4, we return to cut-point spaces with special structures. It is shown that a cut-point space with the order topology is uncountable, and is homeomorphic to the real line $\mathbf{R}$ if it has a countable dense subset. Also it is shown that if a cut-point space is a topological group, then it is homeomorphic to $\mathbf{R}$ provided that it is locally connected or locally compact. In chapter 5, the covering dimension of cut-point spaces is studied briefly. It is proved that if a cut-point space is embeddable in $\mathbf{R}^2$, then its covering dimension is equal to 1.

Throughout this thesis, X denotes a topological space and every subset Y of X is equipped with its subspace topology. A point $x \in X$ is said to be closed (resp. open) if $\{x\}$ is closed (resp. open) in X.

Chapter 2

Basic Topological Properties of Cut-Point Spaces

In this chapter a cut-point space is defined and some examples of cut-point spaces are given. It is proved that a cut-point space is infinite, and non-compact. Some other results about the cardinality of cut-point spaces are obtained. A simple characterization of the Khalimsky line (see Example 2.5) is presented too.

Definitions and Examples

2.1 Definitions. Let X be a non-empty connected topological space. A point x in X is said to be a *cut point* of X if $X\backslash\{x\}$ is a disconnected subset of X. A nonempty connected topological space X is said to be a *cut-point space* if every x in X is a cut point of X.

In the following three examples, $\mathbf{R}^2$ is the Euclidean plane with the standard topology.

2.2 Example. The union of n straight lines in $\mathbf{R}^2$ is a cut-point space if and only if either all of them are concurrent or exactly $n-1$ of them are parallel.

2.3 Example. Let $X_1 = \{(x,y) \in \mathbf{R}^2 : x \leq 0 \text{ and } |y| = 1\}$ and let $X_2 = \{(x,y) \in \mathbf{R}^2 : x > 0 \text{ and } y = \sin\frac{1}{x}\}$. Define $X = X_1 \cup X_2$. Then X is a cut-point space. For each $x \in X$, $X\backslash\{x\}$ has exactly two components. This example shows that Exercise 15 b) in [2, Chapter IV, § 2] is not true. A similar example contradicts part c) of this exercise. Also this example contradicts part

(1) of Corollary 6.12 in [9].

A "connected ordered topological space" (COTS) is a connected topological space X with the following property: if Y is a three-point subset of X, there is a y in Y such that Y meets two connected components of $X \backslash \{y\}$ (see [6]). Put $Y = \{(0,-1),(1,\sin 1),(0,1)\}$ in Example 2.3 to see that X is not a COTS.

2.4 Example. Let $X_0 = \{(x,0) \in \mathbf{R}^2 : x \leq 0\} \cup \{(x,1) \in \mathbf{R}^2 : x > 0\}$ and let for each positive integer n, $Y_n = \{(\frac{1}{n}, y) \in \mathbf{R}^2 : 0 < y \leq 1\}$. Define $X = X_0 \cup (\bigcup_{n=1}^{\infty} Y_n)$. Then X is a cut-point space.

A connected topological space is said to have the "connected intersection property" if the intersection of every two connected subsets of it is connected. In Example 2.4, let $X_1 = X_0 \cup (\bigcup_{n=1}^{\infty} Y_{2n-1})$ and $X_2 = X_0 \cup (\bigcup_{n=1}^{\infty} Y_{2n})$. Since $X_1 \cap X_2 = X_0$ is not connected, X does not possess the connected intersection property. Example 2.4 is a slightly modified version of an example in [11].

2.5 Example (The Khalimsky line). Let $\mathbf{Z}$ be the set of integers and let

$$\mathcal{B} = \{\{2i-1, 2i, 2i+1\} : i \in \mathbf{Z}\} \cup \{\{2i+1\} : i \in \mathbf{Z}\}.$$

Then $\mathcal{B}$ is a base for a topology on $\mathbf{Z}$. The set of integers $\mathbf{Z}$ with this topology is a cut-point space and is called the *Khalimsky line*. Each point in $\mathbf{Z}$ has a smallest open neighborhood and the base $\mathcal{B}$ is the collection of all such neighborhoods. It can be easily seen that the Khalimsky line is irreducible in the sense that no proper subset of it is a cut-point space.

2.6 Example. Let $\mathcal{T}$ be the standard topology of the real line and let A be a dense subset of $\mathbf{R}$ in the topology $\mathcal{T}$. Let $\mathcal{T}' = \{U \cup (A \cap V) : U \in \mathcal{T}$ and $V \in \mathcal{T}\}$. Then $\mathcal{T}'$ is a topology on $\mathbf{R}$ that is finer than $\mathcal{T}$. Thus for every $x \in \mathbf{R}$, $\mathbf{R}\backslash\{x\}$ is a disconnected subset of $\mathbf{R}$ with the topology $\mathcal{T}'$. To prove that $\mathbf{R}$ is a cut-point space in the topology $\mathcal{T}'$, it is sufficient to show that $\mathbf{R}$ is connected in that topology. Let $\mathbf{R} = [U_1 \cup (A \cap V_1)] \cup [U_2 \cup (A \cap V_2)]$ where U_1, V_1, U_2 and V_2 are open subsets of $\mathbf{R}$ in the topology $\mathcal{T}$. If $[U_1 \cup (A \cap V_1)] \cap [U_2 \cup (A \cap V_2)] = (U_1 \cap U_2) \cup \{A \cap [(V_1 \cap U_2) \cup (U_1 \cap V_2) \cup (V_1 \cap V_2)]\} = \emptyset$, then $U_1 \cap U_2 = V_1 \cap U_2 = U_1 \cap V_2 = V_1 \cap V_2 = \emptyset$. Thus $(U_1 \cup V_1) \cap (U_2 \cup V_2) = (U_1 \cap U_2) \cup (V_1 \cap U_2) \cup (U_1 \cap V_2) \cup (V_1 \cap V_2) = \emptyset$. Since $\mathbf{R} = (U_1 \cup V_1) \cup (U_2 \cup V_2)$ and since $\mathbf{R}$ is connected in the topology $\mathcal{T}$, $U_1 = V_1 = \emptyset$ or $U_2 = V_2 = \emptyset$. Therefore $U_1 \cup (A \cap V_1) = \emptyset$ or $U_2 \cup (A \cap V_2) = \emptyset$. This completes the proof of the connectedness of $\mathbf{R}$ in the topology $\mathcal{T}'$.

Cardinality and Non-Compactness of Cut-Point Spaces

Theorem 2.8 is the key theorem of this section. The main theorem of this section is Theorem 2.14 which implies the non-compactness of cut-point spaces. Notation 2.7 is adopted from [9].

2.7 Notation. Let Y be a topological space. We write $Y = A|B$ to mean A and B are two nonempty subsets of Y such that $Y = A \cup B$ and $A \cap \bar{B} = \bar{A} \cap B = \emptyset$.

2.8 Theorem. Let X be a connected topological space, and let x be a cut point of X such that $X\backslash\{x\} = A|B$. Then $\{x\}$ is open or closed. If $\{x\}$ is open, then A and B are closed; if $\{x\}$ is closed, then A and B are open.

Proof. Since A is both open and closed in $X\backslash\{x\}$, there is an open subset V of X such that $A = V \cap (X\backslash\{x\}) = V\backslash\{x\}$, and there is a closed subset F of X such that $A = F \cap (X\backslash\{x\}) = F\backslash\{x\}$. Thus $A = V\backslash\{x\} = F\backslash\{x\}$. Since the assumption $V = F$ contradicts the connectedness of X, we have $\{x\} = V\backslash F$ or $\{x\} = F\backslash V$. If $\{x\} = V\backslash F$, then $\{x\}$ is open and $A = F$ is closed. If $\{x\} = F\backslash V$, then $\{x\}$ is closed and $A = V$ is open. ∎

2.9 Corollary. Let X be a connected topological space, and let Y be the subset of all cut points of X. Then the following statements are obviously true.

(a) Every nonempty connected subset of Y that is not a singleton, contains at least one closed point.

(b) If $x \in Y$ is open, then every limit point of $\{x\}$ in Y is a closed point. ∎

2.10 Lemma. Let X be a connected topological space and let E be a connected subset of X. If $X\backslash E = A|B$, then $A \cup E$ is connected.

Proof. If $A \cup E$ is not connected, then there are subsets C and D of X such that $A \cup E = C|D$. Without loss of generality, we may assume that $E \subseteq C$. Then $D \subseteq A$. But $\overline{(B \cup C)} \cap D = (\bar{B} \cap D) \cup (\bar{C} \cap D) = \bar{B} \cap D \subseteq \bar{B} \cap A = \emptyset$, and $(B \cup C) \cap \bar{D} = (B \cap \bar{D}) \cup (C \cap \bar{D}) = (B \cap \bar{D}) \subseteq B \cap \bar{A} = \emptyset$. Therefore,