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Triality and Supertriality Structures

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To my dear wife

and

my parents.

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In The Name of Allah, The Most Gracious, The Most Merciful.

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Abstract

In this thesis at first, we write down our triality axioms and introduce the Jordan algebra associated to a triality. We constructed a relation between triality and twisor. Then we write down our supertriality axioms and introduce the Jordan superalgebra associated to a supertriality. We constructed a relation between supertriality and supertwisor. Also we define triality manifolds. It is an open question that what kind of manifolds admit a triality structure. Finally we define the cross product space. It is shown that V is a cross product space if and only if $dim(V) = 3$.

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

Twistor Theory began as a subject in the late 60's with the appearance of Penrose (1967). A more definitive statement of its aims and accomplishments was "Twistor Theory: An Approach to the Quantisation of Fields and Space-time" (Penrose and MacCallum) which appeared in 1973. There is a modest point of view which simply holds that the theory is useful for solving some non-linear equations and the object is to discover which ones, and there is full-blooded strain which holds that the repeated occurrence and usefulness of complex analyticity tells one something fundamental about the physical word.

Superalgebras appeared in the context of algebraic topology and homological algebra, but they endured a new impetuous development due to Physics and attempt to capture "the supersymmetry" between Bosons and Fermions. Later, superalgebras proved to be important as purely algebraic objects, they have produced new ideas and methods and have helped to solve some old algebraic problems.

In this thesis we find a relation between triality and twistor theory and we constructed similar relation between supertriality and supertwistor.

1.2 Introduction to Superalgebra

1.2.1 The Rule of Signs

Definition 1.2.1. *A linear space M is called a superspace or $\mathbb{Z}_2$-graded vector space if it admit a decomposition*

$$M = M_0 \oplus M_1.$$

An element of M is called homogeneous of degree ε if $a \in M_\varepsilon$; we write $\widetilde{a} = \varepsilon \in \mathbb{Z}_2$. The elements of M_0 are called even, and those of M_1 are called odd.

A subsuperspace is a subspace $L \subseteq M$ such that

$$L = (L \cap M_0) \oplus (L \cap M_1).$$

If M and N are superspace, we make $M \oplus N$ and $M \otimes N$ into superspace by setting

$$(M \oplus N)_i = M_i \oplus N_i, \qquad (M \otimes N)_i = \bigoplus_{m+n=i} M_m \otimes N_n.$$

Definition 1.2.2. *A superspace A is called superalgebra if A to be compatible the grading*

$$\widetilde{ab} = \widetilde{a} + \widetilde{b}.$$

A superalgera A is called commutative if $ab = (-1)^{\widetilde{a}\widetilde{b}} ba$ for homogeneous $a, b \in A$.

The superidentity is obtained from the corresponding identity following the KASPLANSLY RULE: If two homogeneous adjacent variables a, b are exchanged, then the corresponding term is multiplied by $(-1)^{\widetilde{a}\widetilde{b}}$.

Definition 1.2.3. *The tensor product of two superalgebras A and B is the superspace $A \otimes B$ together with the structure of superalgebras given by*

$$(x \otimes y)(z \otimes t) = (-1)^{\widetilde{y}\widetilde{z}} xz \otimes yt,$$

where $x, y \in A$ and $z, t \in B$.

1.2.2 Modules over superalgebras

Definition 1.2.4. *Let A be a superalgebra. A left module over A is a superspace M together with a left action of A on M, bilinear over $\mathbb{R}$ or $\mathbb{C}$,*

$$A \times M \longrightarrow M,$$

$$(a, m) \to am \qquad \widetilde{am} = \widetilde{a} + \widetilde{m},$$

such that

$$a(bm) = (ab)m.$$

A linear mapping $f : N \longrightarrow M$ of left A-modules is called even if

$$\widetilde{f(n)} = \widetilde{n}$$

$$f(an) = af(n) \qquad a \in A, n \in N.$$

It called odd if

$$\widetilde{f(n)} = \widetilde{n} + 1$$

$$f(an) = (-1)^{\widetilde{a}} af(n).$$

Let $Hom_A(N, M)$ be the superspace of A-homomorphisms

$$Hom_A(N, M) = Hom_A(N, M)_0 \oplus Hom_A(N, M)_1.$$

Definition 1.2.5. *Let M be an A-module, and define ΠM by*

a) $(\Pi M)_\varepsilon = M_{\varepsilon+1}$.

b) Addition in ΠM is the same as in M, and $a(\Pi m) = (-1)^{\widetilde{a}}\Pi(am)$.

A free module M of rank $p|q$ is one which is isomorphic to

$$A^{p|q} = A^p \oplus (\Pi A)^q.$$

1.2.3 Matrix algebra

Given an A-module morphism $A^{m|n} \longrightarrow A^{p|q}$ can be regarded, relative to the canonical bases of $A^{m|n}$ and $A^{p|q}$ as a $(p+q) \times (m+n)$ matrix with entries in A,

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix},$$

which acts on column vector in $A^{m|n}$ from the left. The set $M_A[(p+q) \times (m+n)]$ of such matrices can be graded so as to be naturally isomorphic to the $Hom_A(A^{m|n}, A^{p|q})$, by decreeing that:

X is even if X_1 and X_4 have even entries, while X_2 and X_3 have odd entries;

X is odd if X_1 and X_4 have odd entries, while X_2 and X_3 have even entries.

The set of this form denoted by $M_A(p|q, m|n)$.

Let $f : M \longrightarrow N$ be a homomorphism of free A-module and X the supermatrix corresponding to f. There exist a natural homomorphism $f^* : M^* \longrightarrow N^*$ with the property

$$(f^*(t^*))(s) = (-1)^{\widetilde{ft}} t^*(f(s)),$$

for all $t^* \in N^*$, $s^* \in M^*$.

If $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ then

$$X^{st} = \begin{cases} \begin{pmatrix} X_1^t & X_3^t \\ -X_2^t & X_4^t \end{pmatrix} & \text{for } \widetilde{X} = 0 \\ \begin{pmatrix} X_1^t & -X_3^t \\ X_2^t & X_4^t \end{pmatrix} & \text{for } \widetilde{X} = 1. \end{cases}$$

Here t denotes the usual transpose. A supertranspose has the following properties:

$$(B+C)^{st} = B^{st} + C^{st},$$

$$(BC)^{st} = (-1)^{\widetilde{B}\widetilde{C}} C^{st} B^{st}.$$

We know $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ is invertible if X_1 and X_4 are invertible.

Definition 1.2.6. *For* $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ *we define*

$$Ber X = det(X_1 - X_2 X_4^{-1} X_3)(det X_4)^{-1}.$$

And

$$strac(X) = \begin{cases} trac(X_1) - trac(X_4) & for \quad \widetilde{X} = 0 \\ trac(X_1) + trac(X_4) & for \quad \widetilde{X} = 1. \end{cases}$$

Definition 1.2.7. *Let A ba a commutative algebra, and let M_1 and M_2 be A-modules. An even or odd morphism of A-modules $g : M_1 \otimes M_2 \longrightarrow A$ is called, respectively, an even or odd bilinear form. A bilinear form can be uniquely identified with the $g(m_1, m_2) = g(m_1 \otimes m_2)$, which satisfies the following conditions;*

a) g is biadditive and homogeneous;

b) $g(am_1, m_2) = (-1)^{\widetilde{a}\widetilde{g}} ag(m_1, m_2)$, and $g(m_1, m_2 a) = g(m_1, m_2)a$, for $a \in A$, $m_i \in M_i$.

We associate to an element $m_1 \in M_1$ the mapping $m_2 \mapsto g(m_1, m_2)$, so we obtain a morphism $M_1 \longrightarrow (M_2)^*$ of the same degree as g. Usually , we will denote this morphism *by the same letter* g. *The form is said to be nondegenerate if* $g : M_1 \longrightarrow (M_2)^*$ *is an isomorphism.*

Let $M_1 = M_2 = M$, we say that a form is symmetric if $g(m,n) = (-1)^{\tilde{m}\tilde{n}} g(n,m)$.

1.2.4 Jordan algebra and Jordan superalgebra

Jordan algebra was initially introduced by Jordan (1933) and further developed by Jordan et al (1934). Jordan algebra was used for solving early problem of quantum theory.

Definition 1.2.8. *A Jordan algebra is an algebra if it satisfies:*

1. *Commutativity* $a.b = b.a$
2. *Jordan identity* $a^2.(b.a) = (a^2.b).a$

or equivalently the linearization:

$$(a.b).(c.d) + (a.c).(b.d) + (a.d).(b.c) = ((a.b).c).d + ((a.d).c).b + ((b.d).c).a.$$

Definition 1.2.9. $J = J_{\bar{0}} + J_{\bar{1}}$ *is a Jordan superalgebra if it satisfies:*

1. *Supercommutativity* $a.b = (-1)^{\tilde{a}\tilde{b}} b.a$
2. *Super Jordan identity*

$(a.b).(c.d) + (-1)^{\tilde{b}\tilde{c}}(a.c).(b.d) + (-1)^{\tilde{b}\tilde{d}+\tilde{c}\tilde{d}}(a.d).(b.c) = ((a.b).c).d + (-1)^{\tilde{c}\tilde{d}+\tilde{b}\tilde{c}}((a.d).c).b + (-1)^{\tilde{a}\tilde{b}+\tilde{a}\tilde{c}+\tilde{c}\tilde{d}}((b.d).c).a.$

1.3 Semi-Riemannian Geometry

We begin with a brief introduction to semi-Riemmanian geometry.

1.3.1 Manifolds

Let M^n be a smooth n-dimensional manifold. Hence, M^n is a topological space (Hausdorff, second countable), together with a collection of coordinate charts $(U, x) = (U, x^1, \ldots, x^n)$ (U open in M) covering M such that on overlapping charts (U, x), (V, y), $U \cap V \neq \emptyset$, the coordinates are smoothly related

$$y^i = f^i(x^1, \ldots, x^n), \qquad f^i \in C^\infty, \qquad i = 1, \ldots, n.$$

For any $p \in M$, let T_pM denote the tangent space of M at p. Thus, T_pM is the collection of tangent vectors to M at p. Formally, each tangent vector $X \in T_pM$ is a derivation acting on real valued functions f, defined and smooth in a neighborhood of p. Hence, for $X \in T_pM$, $X(f) \in R$ represents the directional deriviative of f at p in the direction X.

If p is in the chart (U, x) then the coordinate vectors based at p,

$$\frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p, \ldots, \frac{\partial}{\partial x^n}|_p$$

form a basis for T_pM. I.e., each vector $X \in T_pM$ can be expressed uniquely as,

$$X = X^i \frac{\partial}{\partial x^n}|_p, \qquad X^i \in \mathbb{R}.$$

Here we have used the Einstein summation convention: If, in a coordinate chart, an index appears repeated, once up and once down, then summation over that index is implied.

<u>Note</u>: We will sometimes use the shorthand: $\partial_i = \frac{\partial}{\partial x^i}$.

The tangent bundle of M, denoted TM is, as a set, the collection of all tangent vectors,

$$TM = \bigcup_{p \in M} T_pM.$$

To each vector $V \in TM$, there is a natural way to assign to it $2n$ coordinates,

$$V \sim (x^1, \ldots, x^n, V^1, \ldots, V^n),$$

where $(x^1, \ldots, x^n)$ are the coordinates of the point p at which V is based, and $(V^1, \ldots, V^n)$ are the components of V with respect to the coordinate basis vectors $\frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p, \ldots, \frac{\partial}{\partial x^n}|_p$. By this correspondence one sees that TM forms in a natural way a smooth manifold of dimension $2n$. Moreover, with respect to this manifold structure, the natural projection map $\pi : TM \to M$, $V_p \mapsto p$, is smooth.

1.3.2 Lie algebra

A Lie algebra is a real vector space V endowed with a bilinear map $V \times V \longrightarrow V$, denoted by $(x, y) \mapsto [x, y]$ and called bracket of X and Y, satisfying the following two properties for all $x, y, z \in V$:

- Antisymmetry: $[x, y] = -[x, y]$.
- Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

1.3.3 Vector fields

A vector field X on M is an assignment to each $p \in M$ of a vector $X_p \in T_pM$,

$$p \in M \to X_p \in T_pM.$$

If (U, x) is a coordinate chart on M then for each $p \in U$ we have

$$X_p = X^i(p)\frac{\partial}{\partial x^i}|_p.$$

This defines n functions $X^i : U \to \mathbb{R}$, $i = 1, \ldots, n$, the components of X on (U, x). If for a set of charts (U, x) covering M the components X^i are smooth ($X^i \in C^\infty(U)$) then we say that X is a smooth vector field.

Let $\mathfrak{X}(M)$ denote the set of smooth vector fields on M. Vector fields can be added pointwise and multiplied by functions; for $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$,

$$(X+Y)_p = X_p + Y_p, \qquad (fX)_p = f(p)X_p.$$

From these operations we see that $\mathfrak{X}(M)$ is a module over $C^\infty(M)$.

Given $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, X acts on f to produce a function $X(f) \in C^\infty(M)$, defined by,

$$X(f)(p) = X_p(f).$$

With respect to a coordinate chart (U, x) , $X(f)$ is given by,

$$X(f) = X^i \frac{\partial f}{\partial x^i}.$$

Thus, a smooth vector field $X \in \mathfrak{X}(M)$ may be viewed as a map

$$X : C^\infty(M) \to C^\infty(M), \qquad f \mapsto X(f)$$

that satisfies,

(1) $X(af + bg) = aX(f) + bX(g) \qquad (a, b \in \mathbb{R})$,

(2) $X(fg) = X(f)g + fX(g)$.

Indeed, these properties completely characterize smooth vector fields.

Given $X, Y \in \mathfrak{X}(M)$, the Lie bracket $[X, Y]$ of X and Y is the vector field defined by

$$[X,Y] : C^\infty(M) \to C^\infty(M), \qquad [X,Y] = XY - YX,$$

i.e.

$$[X,Y](f) = X(Y(f)) - Y(X(f)).$$

With respect to a coordinate chart, $[X, Y]$ is given by

$$[X,Y] = (X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}) \frac{\partial}{\partial x^j}$$

$$= (X(Y^j) - Y(X^j))\frac{\partial}{\partial x^j}.$$

It is clear from the definition that the Lie bracket is skew-symmetric,

$$[X, Y] = -[Y, X].$$

In addition, the Lie bracket is linear in each slot over the reals, and satisfies,

(1) For all $f, g \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$,

$$[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X.$$

(2) (Jacobi identity) For all $X, Y, Z \in \mathfrak{X}(M)$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

1.3.4 Semi-Riemannian manifolds

Let V be an n-dimensional vector space over $\mathbb{R}$. A symmetric bilinear form $b : V \times V \to \mathbb{R}$ is

(1) positive definite provided $b(v, v) > 0$ for all $v \neq 0$,

(2) nondegenerate provided for each $v \neq 0$, there exists $w \in V$ such that $b(v, w) \neq 0$ (i.e., the only vector orthogonal to all vectors is the zero vector).

Note: 'Positive definite' implies 'nondegenerate'.

A *scalar product* on V is a nondegenerate symmetric bilinear form $\langle , \rangle : V \times V \to \mathbb{R}$. A scalar product space is a vector space V equipped with a scalar product $\langle , \rangle$. Let V be a scalar product space. An orthonormal basis for V is a basis $e_1, \ldots, e_n$ satisfying,

$$\langle e_i, e_j \rangle = \begin{cases} 0 & i \neq j \\ \pm 1 & i = j, \end{cases}$$

or in terms of the Kronecker delta,

$$\langle e_i, e_j \rangle = \varepsilon_i \delta_{ij} \qquad \text{(no sum)}$$

where $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$.

Note: Every scalar product space $(V, \langle , \rangle)$ admits an orthonormal basis.

The signature of an orthonormal basis is the n-tuple $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. It is customary to order the basis so that the minus signs come first. The index of the scalar product space is the number of minus signs in the signature. It can be shown that the index is well-defined, i.e., does not depend on the choice of basis. The cases of most importance are the case of index 0 and index 1, which lead to Riemannian geometry and Lorentzian geometry, respectively.

Definition 1.3.1. *Let M^n be a smooth manifold. A semi-Riemannian metric $\langle , \rangle$ on a M is a smooth assignment to each $p \in M$ of a scalar product $\langle , \rangle_p$ on T_pM,*

$$p \to \langle , \rangle_p : T_pM \times T_pM \to \mathbb{R}.$$

such that the index of $\langle , \rangle_p$ is the same for all p.

By 'smooth assignment' we mean that for all $X, Y \in \mathfrak{X}(M)$, the function $\langle X, Y \rangle$, $p \to \langle X_p, Y_p \rangle_p$, is smooth.

Note: We shall also use the letter g to denote the metric, $g = \langle , \rangle$.

Definition 1.3.2. *A semi-Riemannian manifold is a manifold M^n equipped with a semi-Riemannian metric $\langle , \rangle$. If $\langle , \rangle$ has index 0 then M is called a Riemannian manifold. If $\langle , \rangle$ has index 1 then M is called a Lorentzian manifold.*