IN THE NAME OF GOD

SHIFT OPERATORS ON BANACH SPACES

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In the memory of my mother,

to my wife

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Praise be to God who taught man what he did not know. I would like to express my gratitude to my family and my wife's family without whose help I would have never been able to continue my studies.

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ABSTRACT

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Our major aim in this dissertation is to study and to discuss the results obtained in [8]. An operator T on a Banach space X is called a shift if: (i) T is an injective. (ii) The range of T has codimension 1. (iii) $\bigcap_{i=1}^{\infty} T^i(X) = \{0\}$.

J.R. Holub has obtained several results for shift operators on $C(X)$. We will investigate some questions of Holub and we will obtain extensions of many of his results. In particular, we will show that $C(X, \mathbb{R})$ does not admit a shift operator if X has only countably many components and each component is infinite [Chapter 2].

We will also prove that $C(X, \mathbb{C})$ does not admit a shift operator for certain compact Hausdorff space X. Beside we will show that there exists a compact Hausdorff space X which is not totally disconnected and both $C(X, \mathbb{C})$ and $C(X, \mathbb{R})$ admit shift operators. If $1 \leq p < \infty$ and $(X, \sum, \mu)$ is a σ-finite non-atomic measure space then $L^p_R(\mu)$ does not admit a disjointness preserving shift operator. We will also see that ℓ^p for $1 \leq p \leq \infty$ is the only $L^p_R(\mu)$ space which admits a disjointness preserving shift operator.

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INTRODUCTION

1.1. PRELIMINARIES

1.1.1. Definition. The *support* of a complex (or real) function f on a topological space X is the closure of the set $\{x : f(x) \neq 0\}$. The collection of all continuous complex (or real) functions on X whose support is compact is denoted by $C_c(X)$.

1.1.2. Definition. Let X and Y be two topological spaces; let $q : X \longrightarrow Y$ be a surjective map. The map q is said to be a *quotient map*, provided a subset U of Y is open in Y if and only if $q^{-1}(U)$ is open in X.

1.1.3. Definition. If X is a space and A is a set and if $q : X \longrightarrow A$ is a surjective map then there exists exactly one topology τ on A relative to which q is a quotient map; it is called the *quotient topology* induced by q.

1.1.4. Definition. Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $q : X \longrightarrow X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by q, the space X^* is called a *quotient space* of X.

1.1.5. Definition. Let X be a locally compact Hausdorff space. Take some object out side X, denoted by the symbol ∞ for convenience, and adjoin it to X, forming the set $Y = X \cup \{\infty\}$. Topologize Y by defining the collection of open sets in Y to be all sets of the following types:

(1) U, where U is an open subset of X,

(2) $Y - C$, where C is a compact subset of X.

The space Y is called the *one − point compactification* of X.

1.1.6. Theorem. Let X be a locally compact Hausdorff space which is not compact; let Y be the one-point compactification of X. Then Y is a compact Hausdorff space; X is a subspace of Y; the set $Y - X$ consists of a single point; and $\overline{X} = Y$.

Proof: See [18, Chap. 3, 8.1].

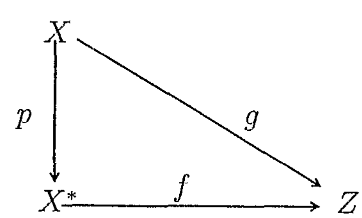
1.1.7. Theorem. Let $g : X \longrightarrow Z$ be a surjective continuous map. Let X^* be the following collection of subsets of X:

$$X^* = \{g^{-1}(z) \ : \ z \in Z\}.$$

Give X^* the quotient topology.

(a) If Z is Hausdorff, so is X^*.

(b) The map g induces a bijective continuous map $f : X^* \longrightarrow Z$, which is a homeomorphism if and only if g is a quotient map.



(p is a projection map.)

Proof: See [18, Chap. 2, 11.2].

1.1.8. Theorem. (Stone-Čech compactification). If X is completely regular, then there is a compact space βX such that:

(a) there is continuous map $\Delta : X \longrightarrow \beta X$ with the property that $\Delta : X \longrightarrow \Delta(X)$ is a homeomorphism;

(b) $\Delta(X)$ is dense in βX;

(c) if $f \in C_b(X)$, where $C_b(X)$ is the space of all bounded continuous real functions, then there is a continuous map $f^\beta : \beta X \longrightarrow F$ such that $f^\beta \circ \Delta = f$.

Moreover, if Ω is a compact space having these properties, then Ω is homeomorphic to βX.

Proof: See [6, Chap. 5, 6.2].

1.1.9. Definition. The compact set βX obtained in the preceding theorem is called the *Stone-Čech compactification* of X. By properties (a) and (b), X can be considered as a dense subset of βX and the map Δ can be taken to be the inclusion map. With this convention, (c) can be interpreted as saying that every bounded continuous function on X has a continuous extension to βX.

1.1.10. Theorem. Let X be compact Hausdorff; let $x \in X$. The intersection of all those sets A containing x which are both open and closed in X equals the component of X containing x.

Proof: See [18, p.235].

1.1.11. Theorem. (Extension property.) Let X be a completely regular Hausdorff space. If Y is a compact Hausdorff space and $g : X \longrightarrow Y$ is a continuous mapping, then g extends uniquely to a continuous mapping from the Stone-$\check{C}$ech compactification βX to Y.

Proof: See [2, 2.73].

1.1.12. Definition. Let X be a topological space. Then X is called *extremely disconnected* if the closure of each open set is open (as well as closed).

At the end of this section we bring some theorems of measure theory which are needed throughout this dissertation.

1.1.13. Theorem. Let μ be a regular Borel measure on a Hausdorff locally compact topological space X. Then there exists a unique closed subset E of X with the following two properties:

(1) $\mu(E^c) = 0$; and

(2) if V is an open set such that $E \cap V \neq \emptyset$, then $\mu(E \cap V) > 0$.

Proof: See [4, p.210].

1.1.14. Definition. The unique set E determined by Theorem 1.1.13 is called the support of μ and is denoted by *supp* μ.

1.1.15. Theorem. Let μ be a signed measure on Σ. Then for every $E \in \Sigma$ the following formulas holds:

(1) $\mu^+(E) = \sup\{\mu(F) \ : \ F \in \Sigma \text{ and } F \subset E\}$.

(2) $\mu^-(E) = \sup\{-\mu(F) \ : \ F \in \Sigma \text{ and } F \subset E\}$.

(3) $|\mu|(E) = \sup\{\sum |\mu(F_i)| \ : \ \{F_i\}$ is a finite disjoint collection of Σ with $\bigcup F_i \subset E\}$.

(4) $|\mu(A)| \leq |\mu|(A)$ for each $A \in \Sigma$.

(5) $\mu = \mu^+ - \mu^-$.

(6) $|\mu| = \mu^+ + \mu^-$.

(7) $\mu^- = (-\mu)^+$.

Proof: See [4, p.229 and p.230].

1.2. DESCRIPTION AND ELEMENTARY PROPERTIES OF VECTOR LATTICES

1.2.1. Definition. A relation $\geq$ on a non-empty set E is called an *order relation* if it satisfies these properties:

(1) $f \geq f$ for all $f \in E$ (reflexivity).

(2) If $f \geq g$ and $g \geq f$ then $f = g$ (antisymmetry).

(3) If $f \geq g$ and $g \geq h$, then $f \geq h$ (transitivity).

The symbolism $f \leq g$ is alternative notation for $g \geq f$.

1.2.2. Definition. An *ordered vector space* is a real vector space E equipped with an order relation satisfying the following two conditions:

(4) If $f \geq g$, then $f + h \geq g + h$ for all $h \in E$.

(5) If $f \geq g$, then $\alpha f \geq \alpha g$ for all $\alpha \geq 0$.

An element f of ordered vector space E is called *positive* if $f \geq 0$ holds. The set of all positive elements of E is called *positive cone* of E and denoted by E^+ (or E_+).

1.2.3. Definition. A *vector lattice* (or *Riesz space* or *linear lattice*) E

is an ordered vector space with the additional property that for every two elements $f, g \in E$ the supremum $f \vee g$ and the infimum $f \wedge g$ exist in E. We remind the reader that two elements $f, g \in E$ have a supremum h in E if $h \geq f$ and $h \geq g$ hold and whenever k is an upper bound of $\{f, g\}$, then $k \geq h$ holds. Clearly $h = f \vee g$ is uniquely determined. In other words, $f \vee g$ is the smallest upper bound of the set $\{f, g\}$. The definition of $f \wedge g$ is similar.

If we index $A = \{f_i \ : \ i \in I\}$, then we may employ the standard lattice notation

$$\sup A = \bigvee_{i \in I} f_i \qquad \text{and} \qquad \inf A = \bigwedge_{i \in I} f_i \text{ [2. } p.210].$$

The geometric interpretation of the lattice structure on a vector lattice is shown in Figure 1.1:

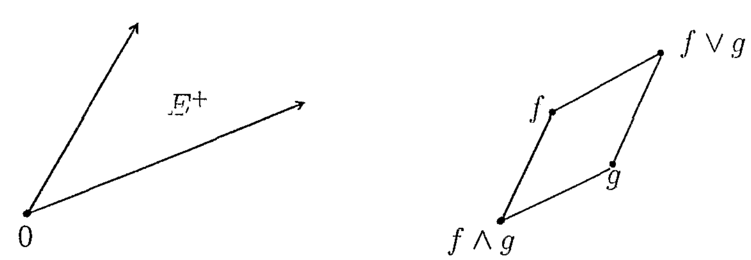


Figure 1.1. The geometry of sup and inf.

A norm $||.||$ on a vector lattice E is said to be a *lattice norm* (or that $||.||$ is compatible with the lattice structure of E), whenever $|f| \leq |g|$ in E implies that $||f|| \leq ||g||$ (where $|f| = f \vee (-f)$).

In this thesis a *normed vector lattice* is a vector lattice that equipped with a lattice norm.