IN THE NAME OF GOD

ON THE SPECTRUM OF A MODULE OVER A COMMUTATIVE RING

BY
MEHRI MOBINI

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.........H. Sharif *H. SHARIF, ph. D., PROF. OF MATHEMATICS (CHAIRMAN)*

M. Ershad... *M. ERSHAD, ph.D., ASSISTANT PROF. OF MATHEMATICS*

M. H. Hashemi... *M. HAKIM HASHEMI, ph. D., ASSISTANT PROF. OF MATHEMATICS*

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I dedicate this dissertation

to my

dear parents

and

my husband for their help

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Mehri Mobini

ABSTRACT

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Let M be a module over a commutative ring R . A submodule K of M is called prime if K≠M and whenever r∈R and m∈M satisfy rm∈K then r∈(K:M) or m∈M, where (K:M)= {r∈R:r M⊆K}. Clearly this is a generalization of the notion of prime ideals of rings .

We shall investigate when the spectrum of M , denoted by spec(M) consisting of all prime submodules of M has a Zariski topology analogous to that for R.

We shall prove that if R is a one- dimensional Noetherian domain then the R-module M is primeless if and only if M is a torsion divisible R-module. We say that M is primeless if spec(M)= ∅. We shall prove that an R-module M is a top module if and only if every prime submodule of M is extraordinary . We say that M is a top module if spec(M) has a Zariski topology .

Z . A . EL –Bast and P.F . Smith have proved that if M is finitely generated then M is a multiplication module if and only

if M/PM is cyclic for all maximal ideals P of R . We shall prove that if M is finitely generated then M is a multiplication module if and only if M is a top module .

Finally we shall prove that a projective R-module M is a top module if and only if M is locally cyclic . We say M is locally cyclic if M_P *is a cyclic module over the local ring* R_P *for every prime ideal P of R .*

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CHAPTER I

INTRODUCTION

1.1. The Scope of the Dissertation

Throughout this dissertation ,all rings R are commutative with identity and all modules are assumed to be unitary .

Let R be a ring and let M be an R-module . A submodule K of M is called prime if $K \neq M$ and whenever $r \in R$ and $m \in M$ satisfy $rm \in K$ then $r \in (K:M)$ or $m \in K$,where $(K:M)=\{r \in R: rM \subseteq K\}$.

In section 2 of this chapter ,we study some properties of prime submodules of a module .Then in section 3 we define the Zariski topology on spec(R) and extend this notion to modules .

In chapter II we study the properties of primeless modules.An R-module M is called primeless if spec(M) = $\varnothing$,where spec(M) denotes the collection of prime submodules of M . In this chapter we prove that if R is a one – dimensional Noetherian domain then the R–module M is primeless if and only if M is a torsion divisible R-module.

In chapter III we investigate conditions on an R-module under which the sets V(N) satisfy the axiom for the closed sets of a topology on spec(M),where for any submodule N of an R-module M we define V(N) to be the set of all prime submodule of M containing N . Also we prove that if R is a perfect ring then *an R-module is a top module if and only if it is cyclic .Note that, M is said to be a top module if the collection of all subsets V(N) of spec(M) is closed under finite unions.*

Z.A.EL-Bast and P.F.Smith have proved in [2] that if M is finitely generated then M is a multiplication module if and only if M/PM is cyclic for all maximal ideals P of R .In chapter IV, by this argument we prove that if M is finitely generated then M is a multiplication module if and only if M is a top module.

In chapter V, we investigate conditions on an R-module under which top modules are locally cyclic and it is proved that a projective R-module M is a top module if and only if M is locally cyclic . Recall that an R-module M is called locally cyclic if M_P *is a cyclic module over the local ring* R_P *for every prime ideal P of R . In section 2 of chapter V we find conditions on R-modules under which direct sum of top modules is a top module and conclude that* $M=\oplus_{i\in I} M_i$ *is a top module if and only if* M_i $(i\in I)$ *are prime-compatible top modules, i.e. for* $i\neq j$ *in I there does not exist a prime ideal P in R with* $spec_P(M_i)$ *and* $spec_P(M_j)$ *are both non-empty .*

1.2. Prime Submodules

Let M be an R-module. For any submodule N of M we denote the annihilator of M/N by (N:M), i.e. $(N:M)=\{r\in R: rM\subseteq N\}$.

Definition 1.2.1. *Let R be a ring and let M be an R-module. A submodule K of M is called prime if $K \neq M$ and whenever $r \in R$ and $m \in M$ satisfy $rm \in K$ then $r \in (K:M)$ or $m \in K$.*

Clearly, any prime ideal of R is a prime R-submodule of the R-module R.

Example 1.2.2. *The torsion submodule T(M) of M over an integral domain is a prime submodule if $T(M) \neq M$, because if $rm \in T(M)$ for some $0 \neq r \in R$ and some $m \in M$, then there exists $0 \neq r' \in R$ such that $r'rm=0$. Since R is a domain, $r'r \neq 0$ and so $m \in T(M)$. Clearly, if $r=0$ then $r \in (T(M):M)$.* □

Lemma 1.2.3. *A submodule K of an R-module M is prime if and only if $P=(K:M)$ is a prime ideal of R and the (R/P)-module M/K is torsion-free.*

Proof : *Let K be a prime submodule of M. Also suppose that $rr' \in P$ and $r \notin P$, for some $r, r' \in R$. Then $rr'M \subseteq K$ and since $r \notin P$, $r'M \subseteq K$. Thus $r' \in P$ and P is a prime ideal of R. Now we know that M/K is an R/P–module, because $P=Ann(M/K)$. suppose that $(r+P)(m+K)=0$, for some $r+P \in R/P$ and $m+K \in M/K$. Therefore $rm+K=0$ and hence $rm \in K$. Consequently $r \in P$ or $m \in K$, i.e $r+P=0$ or $m+K=0$. Thus the R/P-module M/K is torsion-free. Conversly, we assume that $rm \in K$ and $r \notin P$, where $r \in R$ and $m \in M$. Hence $rm+K=(r+P)(m+K)=0$. Since M/K is a torsion-free R/P-module then $m+K=0$ and so $m \in K$. It follows that K is a prime submodule of M.* □

If K is a prime submodule of M and P = (K:M) then K is called a P–prime submodule of M.

Example 1.2.4. *If R is a simple ring, then every non-zero R-module M of R is torsion-free, since for any $o \neq x \in M$, $ann(x) \neq R$ and hence $ann(x)=0$. Also for any proper submodule N of M, $(N:M)=0$ and since (0) is the only maximal ideal of R, (0) is prime. It follows that a simple ring R has the property that every proper submodule N of M is prime.* □

Corollary 1.2.5. *Let K be any submodule of an R-module M such that (K:M) is a maximal ideal of R. Then K is a prime submodule of M. In particular, mM is a prime submodule of an R-module M for every maximal ideal m of R such that $mM \neq M$.*

Proof: *Since $(K:M) \neq R$ then $K \neq M$ and since $(K:M)=m$ is a maximal ideal of R then R/m is a field and M/K is a vector space over R/m. Now if $\bar{r}\,\bar{x}=0$ and $r \neq 0$, where $\bar{r}=r+m$ for some $r \in R$ and $\bar{x}=x+K$ for some $x \in M$, then $\bar{r}^{-1}\,\bar{r}\,\bar{x}=o$ and so $\bar{x}=o$. Thus M/K is a torsion–free R/m-module. It follows that K is a prime submodule of M by Lemma 1.2.3. Now if for some maximal ideal m of R $mM \neq M$, then it is clear that $(mM:M)=m$. Thus mM is a prime submodule of M.* □

Example 1.2.6. *Every proper subspace of a vector space is prime.*

Proof : *Let V be a vector space over the field F and W be a proper subspace of V. Since $rV=V$ for every $0 \neq r \in F$ then (W:V)*

=0 and since (0) is a maximal ideal of F therefore by Corollary 1.2.5 W is a prime submodule of V. □

***Corollary* 1.2.7.** *Let N be a proper submodule of an R-module M and let m be a maximal ideal of R. Then N is m-prime if and only if $mM \subseteq N$. Consequently, if N is an m-prim submodule of M, then so is every proper submodule of M containing N.*

Proof : The necessity is trivial. Conversely if $mM \subseteq N$ then $m \subseteq (N:M)$ and since $N \neq M$ hence $(N:M) \neq R$ therefore $m=(N:M)$. It follows that N is an m–prime submodule of M by Corollary 1.2.5. □

***Proposition* 1.2.8.** *If N is a maximal submodule of an R-module M, then (N:M) is a maximal ideal of R and N is a prime submodule.*

Proof : Let $(N:M) \subseteq m \subseteq R$, where m is an ideal of R. Since N is a maximal submodule of M hence M/N is a simple R–module. It implies that M/N is cyclic and $M/N=(x+N)R$ for some $x \in M$. Thus $m(M/N)=M/N$ or $m(M/N)=0$. If $m(M/N)=M/N$ then $m(M/N)=(x+N)R$ and hence there exists $r \in m$ and $y+N \in M/N$ $(y \in M)$ such that $x+N=r(y+N)$. On the other hand, $y+N=r'(x+N)$, for some $r' \in R$, therefore $(x-rr'x)+N=(1-rr')(x+N)=0$. It follows that $1-rr' \in Ann(M/N)=(N:M) \subseteq m$. Since $r \in m$, $1=1-rr'+rr' \in m$ so $m=R$. Now if $m(M/N)=0$ then $mM \subseteq N$ and so $m \subseteq (N:M) \subseteq m$. Hence $(N:M)=m$. Therefore

$(N:M)$ is a maximal ideal of R. By Corollary 1.2.5 N is a prime submodule of M. □

Remark 1.2.9. *If m is a maximal ideal of a ring R, then not every m-prime submodule of an R-module M is a maximal submodule. In Example 1.2.6 we can see that (0) is a maximal ideal and all maximal or non-maximal subspaces of vector spaces V are (0) – prime submodule in V.*

Corollary 1.2.10. *If M is a finitely generated module, then every proper submodule of M is contained in a prime submodule.*

Proof : Let N be a proper submodule of M and let A be the set of all submodules of M containing N. A is non-empty, because $N \in A$. By Zorn's Lemma, it can easily be proved that there exists a maximal element L in A. Thus L is a maximal submodule of M and by Proposition 1.2.8 L is a prime submodule of M containing N. □

Definition 1.2.11. *An R-module M is called a multiplication module provided that for every submodule N of M there exists an ideal I of R such that $N=IM$.*

Theorem 1.2.12. *Let M be a non-zero R–module, where $R \neq 0$. If M is a multiplication module, then M has at least one prime submodule.*

Proof : Let $M \neq 0$ and $0 \neq m \in M$. Then $I=\{r \in R \mid rm=0\}$ is a proper ideal of R and hence $I \subseteq P$ for some maximal ideal P of R. If $M=PM$ then since $Rm=AM$, for some ideal A of R, we have $Rm=AM=PAM=PRm=Pm$. Therefore $(1-r)m=0$ for some $r \in P$

and hence $(1\text{-}r) \in I$. Since $I \subseteq P$ then $(1\text{-}r) \in P$ and so $1 \in P$, a contradication . Thus $M \neq PM$. Since $(PM:M)=P$ is a maximal ideal , PM is a prime submodule of M by Corollary 1.2.5 . □

For any R-module M, let spec(M) denotes the collection of all prime submodules of M. Now let H be any R-module .For any prime ideal P of R we define:

$$spec_P(H)=\{L \in spec(H) \mid (L:H)=P\}$$

Lemma 1.2.13 . Let P be a prime ideal of R and let M be an R-module . Let N be any submodule of M and let $K \in spec_P(M)$. Then $K \cap N=N$ or $K \cap N \in spec_P(N)$.

Proof : Let $K \cap N \neq N$. For any $r \in P$ we have $rN \subseteq rM \subseteq K$, also $rN \subseteq N$ then $rN \subseteq K \cap N$. Hence $P \subseteq (K \cap N:N)$. Now suppose that $r \in (K \cap N:N)$ then $rN \subseteq K \cap N \subseteq K$. Since $N \subseteq K$ and K is a prime submodule of M then $r \in P$. Thus $(K \cap N:N)=P$. Let $rx \in K \cap N$, where $r \in R$ and $x \in N$, hence $rx \in K$ and so $r \in P$ or $x \in K$. It follows that $r \in P$ or $x \in K \cap N$.Thus $K \cap N \in spec_P(N)$.

1.3. Zariski Topology on Spec(M)

Recall that spec(R) denotes the collection of all prime ideals of R.

For an ideal A of R we define $V(A)=\{P \in spec(R):A \subseteq P\}$. It can easily be checked that $V(\{0\})=spec(R)$ also

$V(R) = \varnothing$

$V(A) \cup V(B)=V(AB)$

$$\cap_{\lambda\in\Lambda} V(A_\lambda)=V(\Sigma_{\lambda\in\Lambda} A_\lambda)$$

Where A and B and A_λ ($\lambda\in\Lambda$) are ideals of R. Thus the V(A) are the closed sets for a topology on spec(R) , called the Zariski topology.

Now we extend this notion to modules . For any submodule N of R-module M we define V(N) to be the set of all prime submodules of M containing N . Of course , V(M) is just the empty set and V(0) is spec(M) .

Lemma 1.3.1. *For any family of submodules N_i ($i\in I$) of M ,*

$$\cap_{i\in I} V(N_i)=V(\Sigma_{i\in I} N_i)$$

Proof : *Let $L\in\cap_{i\in I} V(N_i)$ hence $N_i\subseteq L$ for any $i\in I$ then $\Sigma_{i\in I} N_i\subseteq L$. Conversly, if $L\in V(\Sigma_{i\in I} N_i)$ then $N_i\subseteq\Sigma N_i\subseteq L$ for any $i\in I$, hence $\cap_{i\in I} N_i\subseteq L$.□*

Thus if ε(M) denotes the collection of all subsets V(N) of spec(M) then ε(M) contains the empty set and spec(M) , also ε(M) is closed under arbitrary intersections. Now if ε(M) is closed under finite unions , i.e. for any submodules N and L of M there exists a submodule J of M such that $V(N)\cup V(L)=V(J)$, in this case ε(M) satisfies the axioms for the closed subsets of a topological space and therefore M is a module with Zariski topology or a top module for short.

CHAPTER II

PRIMELESS MODULES

In this chapter we study the properties of primeless modules and prove that if R is a one-dimensional Noetherian domain then an R-module M is primeless if and only if M is a torsion divisible R-module.

For any R-module M, let spec(M) denote the collection of prime submodules of M. Recall that for any ring R, it is known that $R \neq (0)$ if and only if $spec(R) \neq \varnothing$. However we can see in this chapter that for a module M it is not always true that if $M \neq (0)$ then $spec(M) \neq \varnothing$. We call such modules M primeless.

For example the group $\mathbf{Z}(p^{\infty})$ is a primeless $\mathbf{Z}$-module. To see why this is the case, first we define $G_n = \{\alpha \in \mathbf{Q}/\mathbf{Z} : \alpha = r/p^n + \mathbf{Z}$, for some $r \in \mathbf{Z}\}$ $(n \in N \cup \{0\})$. It can easily be checked that for any $n \in N \cup \{0\}$, G_n is a submodule of $\mathbf{Z}(p^{\infty})$ and all the proper submodules of $\mathbf{Z}(p^{\infty})$ are G_n's$(n \in N \cup \{0\})$. We claim that $(G_n : \mathbf{Z}(p^{\infty})) = 0$ for every $n \in N \cup \{0\}$. Suppouse that $(G_n : \mathbf{Z}(p^{\infty})) \neq (0)$ for some $n \in N \cup \{0\}$ and let $0 \neq r \in (G_n : \mathbf{Z}(p^{\infty}))$. Put $r = p^t a$, where $a \in Z$ and t is the largest integer in $N \cup \{0\}$ such that p^t divides r. Let $\alpha' = 1/p^{t+n+1} + \mathbf{Z}$. Then $r\alpha' = (p^t a)/p^{t+n+1} + \mathbf{Z} = a/p^{n+1} + \mathbf{Z}$ and $r\alpha' \notin G_n$, we have a