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Résolutions et Régularité de Castelnuovo-Mumford

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Dedicated to

my wife

and to

my mother

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Abstract

In this thesis, we study square-free monomial ideals of the polynomial ring $S = K[x_1, \ldots, x_n]$ which have a linear resolution. By remarkable result of Bayer and Stilman [BS] and the technique of polarization, classification of homogeneous ideals with linear resolution is equivalent to classification of square-free monomial ideals with linear resolution. However, classification of square-free monomial ideals with linear resolution seems to be difficult because by Eagon-Reiner Theorem [ER], this is equivalent to classification of Cohen-Macaulay ideals.

It is worth to note that, square-free monomial ideals in S are in one-to-one correspondence to Stanley-Reisener ideals of simplicial complexes on one hand and the circuit ideal of clutters from another hand. This correspondence motivated mathematicians to use the combinatorial and geometrical properties of these objects in order to get the desired algebraic results.

Classification of square-free monomial ideals with 2-linear resolution, was successfully done by Fröberg [Fr] in 1990. Fröberg observed that the circuit ideal of a graph G has a 2-linear resolution if and only if G is chordal, that is, G does not have an induced cycle of length > 3. In [Em, ThVt, VtV, W] the authors have partially generalized the Fröberg's theorem for degree greater than 2. They have introduced several definitions of chordal clutters and proved that, their corresponding circuit ideals have linear resolutions.

Viewing cycles as geometric objects (triangulation of closed curves), in this thesis we try to generalize the concept of cycles to triangulation of pseudo-manifolds and get a partial generalization of Fröberg's theorem for higher dimensional hypergraphs.

All the results in Chapters 4 and 5 and some results in Chapter 3 are devoted to be original.

Keywords: Minimal Free Resolution, Castelnuovo-Mumford Regularity, Clutter, Betti Number, Pseudo-manifold, Triangulation, Simplicial Complex.

Mathematics Subject Classification[2010]: 13D14, 13D02, 13D45, 13F55, 16E05, 51H30.

Résumé

Le sujet de cette thèse, est l'étude d'idéaux monomiaux libres de carrés de l'anneau de polynômes $S = K[x_1, \ldots, x_n]$, qui ont une résolution linéaire. D'après un résultat remarquable de Bayer et Stilman [BS] et en utilisant la polarisation, la classification des idéaux monomiaux ayant une résolution linéaire, est équivalente à la classification des idéaux monomiaux libres de carrés ayant une résolution linéaire. De plus le théorème de Eagon-Reiner, établit une dualité entre les idéaux monomiaux libres de carrés ayant une résolution linéaire et les idéaux monomiaux libres de carrés Cohen-Macaulay, ce qui montre que le problème de classification des idéaux monomiaux libres de carrés ayant une résolution linéaire est très difficile.

Nous rappelons que, les idéaux monomiaux libres de carrés sont en correspondance biunivoque avec les complexes simpliciaux d'une part, et d'autre part avec les clutters. Ces correspondances nous motivent pour utiliser les propriétés combinatoires des complexes simpliciaux et des clutters pour obtenir des résultats algébriques. La classification des idéaux monomiaux libres de carrés ayant une résolution linéaire engendrés en degré 2 a été faite par Fröberg [Fr] en 1990. Fröberg a observé que l'idéal des circuits d'un graphe G a une résolution 2-linéaire si et seulement si, G est un graphe de cordes, i.e. il n'a pas de cycles minimaux de longueur plus grande que 4. Dans [Em, ThVt, VtV, W] les auteurs ont partiellement généralisé les résultats de Fröberg à des idéaux engendrés en degré ≥ 3. Ils ont introduit plusieurs définitions de clutters de cordes et démontré que les idéaux de circuits correspondant ont une résolution linéaire.

Nous pouvons voir les cycles du point de vue topologique, comme la triangulation d'une courbe fermée, dans cette thèse nous utiliserons cette idée pour étudier des clutters associés à des triangulation de pseudo manifolds en vue d'obtenir une généralisation partielle des résultats de Fröberg à des idéaux engendrés en degré ≥ 3. Nous comparons notre travail à ceux de [Em, ThVt, VtV, W]. Nous présentons nos résultats dans le chapitres 4 et 5.

Mots clés: Résolution libre minimale, Régularité de Castelnuovo-Mumford, Idéaux monomiaux, Clutters, Nombres de Betti, Pseudo-manifold, Triangulation.

Classification AMS[2010]: 13D14, 13D02, 13D45, 13F55, 16E05, 51H30.

Contents

Introduction

Castelnuovo-Mumford regularity is one of the most fundamental invariants in Commutative Algebra and Algebraic Geometry. One of its first hidden appearances may be found in Castelnuovo's work on linear systems on smooth projective space curves [Cas, 1893]. Castelnuovo's result, gives a sharp upper bound on the largest degree r such that, the complete linear system of the r-fold plane sections on the given curve, is not cut out by surfaces of degree r. Although this result is of fairly geometric appearance, Castelnuovo's method of proof has a rather algebraic flavor.

Another early invisible occurrence of Castelnuovo-Mumford regularity was initiated in the work of Hermann [Her, 1926]. The results of Hermann show that, the minimal free resolution of an ideal generated by finitely many homogeneous polynomials, can be computed in a (finite) number of steps which depends only on the number of indeterminates of the ambient ring and the maximal degree of the given polynomials.

Hermann's work is not at all constructive, and so it does not give rise to an explicit algorithm. It was indeed only around 1980, when such algorithms became practicable, based on Gröbner base techniques, implemented in Computer Algebra Systems like Macaulay, CoCoA, Singular and powered by high performance computers. And indeed:

> Castelnuovo-Mumford regularity provides the ultimate bound of complexity for these algorithms.

D. Bayer and M. Stilman [BS2] showed that, an estimate of the regularity of an ideal, gives a bound on complexity of algorithms for computing syzygies.

In 1966, Mumford gave a first proper definition of Castelnuovo-Mumford regularity (see [M]), which he called Castelnuovo regularity. In fact, Mumford did define the notion of being m-regular in the sense of Castelnuovo for a coherent sheaf of ideals over a projective space and a given integer m. More precisely, a sheaf of ideals over a projective space is called m-regular, if for all positive values of i, the i-th Serre cohomology group of the $(m-i)$-fold twist of this sheaf vanishes. The minimal possible value of m is what today usually is called the Castelnuovo-Mumford regularity of the sheaf of ideals in question. Moreover, Mumford did prove a fundamental bounding result, namely:

> The Castelnuovo-Mumford regularity of a coherent sheaf of ideals over a projective space is bounded by the Hilbert polynomial of this ideal.

In fact Mumford's arguments allow to make this bound explicit. Although Castelnuovo-Mumford regularity was originally defined in terms of sheaf cohomology, it may be expressed in terms of degrees of syzygies and hence is of basic significance in classical Projective Algebraic Geometry.

Castelnuovo-Mumford regularity also found much interest in Commutative Algebra. In 1984, D. Eisenbud and S. Goto [EG] made explicit the link between this algebraic Castelnuovo-Mumford regularity of a graded module over a polynomial ring and its minimal free resolution.

One of the aspects that makes the regularity very interesting, is that Castelnuovo-Mumford regularity can be computed in different ways. In pure algebraic setting, it is defined as follows:

Definition 0.0.1. Let K be a field and let S be a polynomial ring over K. Let $M = \oplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded S-module. In most interesting case, M is an ideal of S. For every $i \in \mathbb{N}\cup\{0\}$, one defines:

$$t_i^S(M) = \max\{j: \quad \beta_{i,j}^K(M) \neq 0\}$$

where $\beta_{i,j}^K(M)$ is the i,j-th graded Betti number of M as an S-module, i.e.

$$\beta_{i,j}^K(M) = \dim_K \operatorname{Tor}_i^S(K, M)_j$$

and $t_i^S(M) = -\infty$, if it happens that $\operatorname{Tor}_i^S(K, M) = 0$.

The *Castelnuovo-Mumford regularity* of M, $\operatorname{reg}(M)$, is given by:

$$\operatorname{reg}(M) = \sup\{t_i^S(M) - i: \quad i \in \mathbb{Z}\}.$$

We say that M has a d-linear resolution, if M is generated by homogeneous elements of degree d and $\operatorname{reg}(I) = d$. That is, the graded minimal free resolution of I is of the form:

$$0 \longrightarrow S^{\beta_s}(-d-s) \longrightarrow \cdots \longrightarrow S^{\beta_1}(-d-1) \longrightarrow S^{\beta_0}(-d) \longrightarrow I \longrightarrow 0.$$

Among all the interesting problems in Castelnuovo-Mumford regularity, classification of ideals with linear resolution is of great importance. Proving that a class of ideals has a d-linear resolution, is difficult in general. However, some classes of ideals with linear resolution may be found in [AHH, AHH2, ANH, CoH, Em, EO, EOS, Fr, HHZ, ThVt, Mo, MNYZ, VtV, W, Zh].

Classification of square-free monomial ideals with 2-linear resolution, was successfully done by Fröberg [Fr] in 1990. Fröberg observed that, the circuit ideal of a graph G has a 2-linear resolution if and only if G is chordal, that is, G does not have an induced cycle of length > 3.

Theorem 0.0.2 (Fröberg's Theorem [Fr])**.** *Let $G \neq \mathcal{C}_{n,2}$ be a graph on vertex set $[n]$ and $I = I(\bar{G})$ be the circuit ideal of G. The ideal I has a 2-linear resolution if and only if G is chordal.*

Fröberg's Theorem in particular implies that, having 2-linear resolution does not depend on the characteristic of the base field K. However, in general having linear resolution does depend on the characteristic of the base field (see for instance, Example 5.3.19).

Trying to generalize Fröberg's result for square-free ideals generated in degree greater than 2, some mathematicians have introduced various definitions of chordal hypergraphs and they proved that the corresponding circuit ideals have a linear resolution over any field K (see for example [Em, ThVt, VtV, W]).

In this thesis, we study square-free monomial ideal with linear resolution. Our method in this thesis, is to look at square-free monomial ideals in both aspects of combinatorics and geometrics.

In Chapter 1, we review basic notations and definitions concerning graded minimal free resolution, depth of module and local cohomology. The notions and remarks in this chapter are essential for the remaining of this thesis.

Chapter 2 of this thesis is devoted to introduce the main subject of this thesis. In this chapter, first we introduce the definition of Castelnuovo-Mumford regularity and then we study the basic properties of Castelnuovo-Mumford regularity. As it is mentioned before, the goal of this thesis is to study square-free monomial ideals with linear resolution, that is, the ideals with generators consist of square-free monomials of degree d and its Castelnuovo-Mumford regularity is again d. In Section 2.2, we will present a survey of known results on ideals with linear resolution.

Chapter 3 of this thesis is in fact the language of this thesis. First in Section 3.1, we outline that the classification of *homogeneous* ideal of $S =$

$K[x_1, \ldots, x_n]$ with linear resolution is equivalent to classification of *square-free* monomial ideal with linear resolution. That is why, in the remaining of this thesis, we consider only square-free monomial ideals of the polynomial ring $S = K[x_1, \ldots, x_n]$.

In Section 3.2, we deal with the notions of simplicial complexes. With a simplicial complex Δ, one can associate a square-free monomial ideals I_Δ whose generators correspond to the non-faces of Δ. This ideal is called Stanley-Reisner ideal of Δ. Note that, there exists a bijection between square-free monomial ideal $I \subset K[x_1, \ldots, x_n]$ and simplicial complexes Δ on vertex set $[n]$, given by $\Delta \leftrightarrow I_\Delta$. This correspondence, motivated us to investigate the interaction between the homological properties of these objects and algebraic properties of square-free monomial ideals.

Alexander duality plays an important role in study of minimal free resolution of Stanley-Reisner ideal. In particular, Eagon and Reiner used Alexander dual complexes and proved the following interesting theorem:

Theorem 0.0.3 (Eagon-Reiner)**.** *Let Δ be a simplicial complex on vertex set $[n]$. The ideal $I_\Delta \subset S = K[x_1, \ldots, x_n]$ has a q-linear resolution if and only if $\Delta^\vee$ is Cohen-Macaulay over K of dimension $n - q$.*

This theorem and Mayer-Vietoris long exact sequence on local cohomologies, will be frequently used in Chapter 4 and play a key role in many proofs.

We recall that, all the generators of an ideal with linear resolution have the same degree. So, it is worth to find a correspondence between square-free monomial ideals generated in same degree with some other combinatorial or geometrical objects rather than simplicial complexes. This leads us to investigate clutters and (pseudo-)manifolds rather than simplicial complexes.

With a d-uniform clutter $\mathcal{C}$ on vertex set $[n]$, we associate a square-free monomial ideal, $I(\bar{\mathcal{C}})$, whose generators are:

$$\left\{ \prod_{i \in F} x_i : \quad F \subset [n],\ |F| = d,\ F \notin \mathcal{C} \right\}.$$

This ideal is called the circuit ideal of $\mathcal{C}$. Note that, we have a bijection between square-free monomial ideal $I \subset K[x_1, \ldots, x_n]$ generated in the same degree d, with d-uniform clutters, given by $\mathcal{C} \leftrightarrow I(\bar{\mathcal{C}})$. Moreover, any triangulation of a manifold or a pseudo-manifold gives rise to a square-free monomial ideal in $K[x_1, \ldots, x_n]$.

The aim of this chapter is to study properties of simplicial complexes, clutters and triangulations.

Chapter 4 is the combinatorial core of this thesis. The goal of this chapter is to do some operations on a given graph (or clutter) to reduce it to a smaller graph (or clutter), such that the Castelnuovo-Mumford regularity of corresponding circuit ideals, does not change under these operations. As consequences of these operations:

- We will find some alternative proofs for Fröberg's Theorem.
- We will find an alternative proof for linearity of circuit ideals of generalized 3-uniform chordal clutter as defined by Emtander.
- We introduce a combinatorial criterion in order to check that if the circuit ideal of a given 3-uniform clutter has a linear resolution.
- We will find a large class of ideals with linear resolution.

Also, we compare several definitions of chordal clutters and some open problems for further studies are given in this chapter.

To attack to the problem of classification of ideas with d-linear resolution, in Chapter 5, we investigate clutters whose their circuit ideals do not have linear resolution, but any proper subclutter of them has a linear resolution. This chapter generalize many of the results in Chapter 4 for arbitrary d-uniform clutters. But the method in this chapter, is not algebraic combinatorics but is algebraic topology.

The circuit ideal of clutters which are minimal to linearity, is contained in the class of square-free monomial ideals I_Δ, with $\operatorname{indeg}(I_\Delta) = 1 + \dim\Delta$. So, first we deal with the class of square-free monomial ideals I_Δ with $\operatorname{indeg}(I_\Delta) \geq 1+\dim\Delta$. The results in this section, enable us to find precisely, the minimal free resolution of ideals which are minimal to linearity. Some nice classes of clutters which are minimal to linearity are pseudo-manifolds, but unfortunately, pseudo-manifolds are strictly contained in this class. However, using the results in this chapter, we can compute the graded Betti numbers of the circuit ideals of an arbitrary pseudo-manifolds (like a triangulation of sphere, projective plane, Klein bottle, etc.).

Also, for two d-uniform clutters $\mathcal{C}_1$ and $\mathcal{C}_2$, we will prove that:

$$\operatorname{reg} I(\overline{\mathcal{C}_1 \cup \mathcal{C}_2}) = \max\{\operatorname{reg} I(\bar{\mathcal{C}}_1),\ \operatorname{reg} I(\bar{\mathcal{C}}_2)\}$$

whenever, $V(\mathcal{C}_1) \cap V(\mathcal{C}_2)$ is a clique or $\mathrm{SC}(\mathcal{C}_1) \cap \mathrm{SC}(\mathcal{C}_2) = \varnothing$. Again, this leads to an alternative proof for Fröberg Theorem as well as linearity of circuit ideal of generalized d-uniform chordal clutters as defined by Emtander.

Finally, in the last section, for a given square-free monomial I generated in degree d, we define a square-free monomial ideal $\hat{I}$, generated in degree $d+1$ which is very closed to I in regularity. In fact we have:

$$\operatorname{reg}(\hat{I}) = \begin{cases} \operatorname{reg}(I), & \text{if } \operatorname{reg}(I) > d; \\ 1 + \operatorname{reg}(I), & \text{if } \operatorname{reg}(I) = d. \end{cases}$$

This enables us to generate a square-free monomial ideal with $(d+1)$-linear resolution from a square-free ideal I with d-linear resolution.

The results in Chapter 5 have many hidden ideas for further studies in this area.

We acknowledge the support provided by the Computer Algebra Systems CoCoA and Singular [CCA, Si] for the extensive experiments which helped us to obtain some of the results in this thesis. Throughout this thesis, all known definitions and statements are quoted with a reference afterwards and all others with no references are supposed to be new. The results in Chapter 4 and Chapter 5 appear(ed) in [MNYZ, MYZ, MYZ2].

Chapter 1

Commutative Algebra

In this chapter, we recall some basic notions and results that will be used later. Throughout this thesis, all rings are considered to be commutative with the identity $1 \neq 0$.

1.1 Graded Modules, Hilbert Series

In commutative algebra, graded rings and graded modules are of great importance, especially local rings. That is, graded rings with just one graded maximal ideal. Graded local rings share many properties with polynomial rings. For example, consider the polynomial ring $S = K[x_1, \ldots, x_n]$ in n variables over a field K; if $n > 0$, this has infinitely many maximal ideals but $(x_1, \ldots, x_n) \subset S$ is the only graded maximal ideal of S.

Definition 1.1.1. A ring A is called *graded* (or more precisely, $\mathbb{Z}$-*graded*), if there exists a family of subgroups $\{A_n\}_{n\in\mathbb{Z}}$ of A such that,

(a) $A = \oplus_n A_n$ (as abelian groups), and

(b) $A_n \cdot A_m \subset A_{n+m}$, for all n, m.

Note that if $A = \oplus_n A_n$ is a graded ring, then A_0 is a subring of A, $1 \in A_0$ and A_n is an A_0-module for all n.

Let A be a ring and $x_1, \ldots, x_n$ be indeterminates over A. For $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$, let $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_n^{m_n}$. Then the polynomial ring $S = A[x_1, \ldots, x_n]$ is a graded ring, where:

$$S_i = \{ \sum_{\mathbf{m}\in\mathbb{N}^n} r_{\mathbf{m}}\mathbf{x}^{\mathbf{m}} : \quad r_{\mathbf{m}} \in A \text{ and } m_1 + \cdots + m_n = i\}.$$

This is called the *standard grading* of the polynomial ring $A[x_1, \ldots, x_n]$.

A product $x_1^{m_1} \cdots x_n^{m_n}$ with $m_i \in \mathbb{N}$ is called a *monomial*. The set of monomials of S, is denoted by $\mathrm{Mon}(S)$. A monomial $x_1^{m_1} \cdots x_n^{m_n}$ is called *square-free monomial*, if $m_i \leq 1$, for $i = 1, \ldots n$.

An ideal $I \subset S$ is called (*square-free*) *monomial ideal*, if it is generated by (square-free) monomials.

Definition 1.1.2. Let A be a graded ring and M an A-module. We say that M is a *graded A-module* (or has an *A-grading*), if there exists a family of subgroups $\{M_n\}_{n\in\mathbb{Z}}$ of M such that,

(a) $M = \oplus_n M_n$ (as abelian groups), and

(b) $A_n \cdot M_m \subset M_{n+m}$, for all n, m.

If $u \in M \setminus \{0\}$ and $u = u_{i_1} + \cdots + u_{i_k}$ where $u_{i_j} \in M_{i_j} \setminus \{0\}$, then $u_{i_1}, \ldots, u_{i_k}$ are called the *homogeneous components* of u.

For a non-zero element $u \in M_i$, the *degree* of u is denoted by $\deg(u)$ which we set to be i.

We let $\mathscr{M}(A)$ be the category of finitely generated graded A-modules. A *homogeneous homomorphism* $\varphi : M \longrightarrow N$ of graded A-modules of degree d is an A-module homomorphism such that $\varphi(M_i) \subset N_{i+d}$, for all i. For example, if $f \in A$ is homogeneous of degree d, then the multiplication map $A(-d) \longrightarrow A$, with $g \mapsto fg$ is a homogeneous homomorphism. Here, for a graded A-module W and an integer a, one denotes by $W(a)$ the graded A-module whose graded components are given by $W(a)_i = W_{a+i}$. One says that, $W(a)$ arises from W by applying the *shift* a. The morphisms being the homogeneous homomorphisms $M \longrightarrow N$ of degree 0, simply called *homogeneous homomorphisms*.

Definition 1.1.3. Let $M = \oplus M_n$ be a graded A-module and N a submodule of M. For each $n \in \mathbb{Z}$, let $N_n = N \cap M_n$. If the family of subgroups $\{N_n\}$ makes N into a graded A-module, we say that N is a *graded* (or *homogeneous*) *submodule* of M. The graded submodules of A is called *homogeneous* (or *graded*) ideal of A.

Note that for any submodule N of M, $A_n \cdot N_m \subset N_{n+m}$. Thus, N is graded if and only if $N = \oplus_n N_n$. In particular, every (square-free) monomial ideal of A is homogeneous ideal.

Proposition 1.1.4. *Let A be a graded ring, M a graded A-module and N a submodule of M. The following statements are equivalent:*

(i) *N is a graded A-module.*

(ii) $N = \sum_n (N \cap M_n)$

(iii) *For every $u \in N$, all the homogeneous components of u are in N.*

(iv) *N has a homogeneous set of generators.*

Definition 1.1.5. Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K with the grading induced by $\deg(x_i) = d_i$, where d_i is a positive integer. If $M = \bigoplus_{i=0}^{\infty} M_i$ is a finitely generated $\mathbb{N}$-graded module over S, its *Hilbert function* and *Hilbert series* are defined by:

$$H(M,i) = \dim_K(M_i) \qquad \text{and} \qquad F(M,t) = \sum_{i=0}^{\infty} H(M,i)t^i.$$

Theorem 1.1.6 (Hilbert-Serre)**.** *Let K be a field and $S = K[x_1, \ldots, x_n]$ a polynomial ring graded by $\deg(x_i) = d_i \in \mathbb{N}^+$. If M is a finitely generated $\mathbb{N}$-graded S-module, then the Hilbert series of M is a rational function that can be written as:*

$$F(M,t) = \frac{h(t)}{\prod_{i=0}^{n}(1-t^{d_i})}, \qquad \textit{for some} \quad h(t) \in \mathbb{Z}[t].$$

In particular, if $d_i = 1$, for all i, then there is a unique polynomial $h(t) \in \mathbb{Z}[t]$ such that:

$$F(M,t) = \frac{h(t)}{(1-t)^d}, \qquad \textit{and} \quad h(1) \neq 0.$$

The number $e(M) = h(1)$ in the above theorem is called the *multiplicity* of the module M.

Definition 1.1.7. Let R be standard graded ring and M be a graded R-module such that,

$$h(t) = h_0 + h_1 t + \cdots + h_r t^r$$

is the (unique) polynomial with integer coefficients such that $h(1) \neq 0, h_r \neq 0$ and satisfying

$$F(M,t) = \frac{h(t)}{(1-t)^d},$$

where, $d = \dim(M)$. The **h**-*vector* of M is defined by $\mathbf{h}(M) = (h_0, \ldots, h_r)$.

Example 1.1.8. If we consider the polynomial ring $S = K[x_1, \ldots, x_n]$ in n variables over the field K and $\deg(x_i) = 1$ (for $i = 1, \ldots, n$), then we have:

$$H(S, i) = \dim_K(S_i) = \binom{i+n-1}{n-1},$$

and for $n \geq 1$,

$$F(M, t) = \sum_{i=0}^{\infty} (\dim_K(S_i))\, t^i = \sum_{i=0}^{\infty} \binom{i+n-1}{n-1} t^i = \frac{1}{(1-t)^n}.$$

1.2 Graded Minimal Free Resolution

Throughout this thesis, we let K be a field, $(R, \mathfrak{m})$ a Noetherian graded local ring with residue field K or a standard graded K-algebra with graded maximal ideal $\mathfrak{m}$. We write S for the polynomial ring $K[x_1, \ldots, x_n]$ with the standard grading.

We let M be a finitely generated R-module and will assume that M is graded, if R is graded.

Now, let M be a finitely generated graded R-module with homogeneous generators $m_1, \ldots, m_r$ and $\deg(m_i) = a_i$, for $i = 1, \ldots, r$. Then, there exists a surjective R-module homomorphism $F_0 = \bigoplus_{i=1}^{r} Re_i \to M$ with $e_i \mapsto m_i$. Assigning to e_i the degree a_i, for $i = 1, \ldots, r$ the map $F_0 \longrightarrow M$ becomes a morphism in $\mathscr{M}(R)$ and F_0 becomes isomorphic to $\oplus_{i=1}^{r} R(-a_i)$. Thus, we obtain the exact sequence:

$$0 \longrightarrow U \longrightarrow \bigoplus_j R(-j)^{\beta_{0,j}^R} \longrightarrow M \longrightarrow 0,$$

where $\beta_{0,j}^R = |\{i : a_i = j\}|$, and where $U = \operatorname{Ker}\left(\oplus_j R(-j)^{\beta_{0,j}^R} \to M\right)$.

The module U is a graded submodule of $F_0 = \oplus_j R(-j)^{\beta_{0,j}^R}$. By Hilbert's basis theorem for modules, we know that U is finitely generated and hence we find again an epimorphism $\oplus_j R(-j)^{\beta_{1,j}^R} \to U$. Composing this epimorphism with the inclusion map $U \to \oplus_j R(-j)^{\beta_{1,j}^R}$, we obtain the exact sequence:

$$\bigoplus_j R(-j)^{\beta_{1,j}^R} \longrightarrow \bigoplus_j R(-j)^{\beta_{0,j}^R} \longrightarrow M \longrightarrow 0$$

of graded R-modules. Proceeding in this way, we obtain a long exact sequence:

$$\mathscr{F} : \quad \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of graded R-modules with $F_i = \oplus_j R(-j)^{\beta^R_{i,j}}$. Such an exact sequence is called a *graded free R-resolution of M.*

It is clear from our construction that, the resolution obtained is by no means unique. On the other hand, if we choose in each step of the resolution a minimal presentation, the resolution will be unique up to isomorphism, as we shall see later.

A set of homogeneous generators $m_1, \ldots, m_r$ of M is called *minimal*, if no proper subset of it generates M.

Lemma 1.2.1. *Let $m_1, \ldots, m_r$ be a homogeneous set of generators of the graded R-module M. Let $F_0 = \oplus_{i=1}^r Re_i$ and let $\varepsilon : F_0 \to M$ be the epimorphism with $e_i \mapsto m_i$, for $i = 1, \ldots, r$. Then, the following conditions are equivalent:*

(i) *$m_1, \ldots, m_r$ is a minimal system of generators of M.*

(ii) $\mathrm{Ker}\,(\varepsilon) \subset \mathfrak{m}F_0$, *where $\mathfrak{m}$ is the unique homogeneous maximal ideal of R.*

Let M be a finitely generated graded R-module. A graded free R-resolution $\mathscr{F}$ of M is called *minimal*, if for all i, the image of $F_{i+1} \to F_i$ is contained in $\mathfrak{m}F_i$. Lemma 1.2.1 implies at once that, each finitely generated graded R-module admits a minimal free resolution.

The next result shows that, the numerical data given by a graded minimal free R-resolution of M depends only on M and not on the particular chosen resolution.

Proposition 1.2.2. *Let M be a finitely generated graded R-module and*

$$\mathscr{F}: \quad \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

a minimal graded free R-resolution of M with $F_i = \oplus_j R(-j)^{\beta^R_{i,j}}$, for all i. Then,

$$\beta^R_{i,j} = \dim_K \mathrm{Tor}^R_i\,(K, M)_j$$

for all i and j.

The numbers $\beta^R_{i,j}$ are called the *graded Betti numbers of M*, and $\beta^R_i = \sum_j \beta^R_{i,j} (= \mathrm{rank} F_i)$ is called the *i-th Betti number of M*. As long as we work with the polynomial ring $S = K[x_1, \ldots, x_n]$, we write $\beta^K_{i,j}$ instead of $\beta^S_{i,j}$. Also, the number

$$\mathrm{projdim}\, M = \sup\{i\colon \quad \mathrm{Tor}^R_i\,(K, M) \neq 0\}.$$

is called the *projective dimension of M*.

We close this section by stating that, not only are the graded Betti numbers determined by a minimal graded free resolution, but also, a minimal graded free resolution of M is unique up to isomorphisms.

Proposition 1.2.3. *Let M be a finitely generated graded R-module and let $\mathscr{F}$ and $\mathscr{G}$ be two minimal graded free R-resolutions of M. Then, the complexes $\mathscr{F}$ and $\mathscr{G}$ are isomorphic.*

Theorem 1.2.4 (Graded Hilbert syzygy theorem, [CLO, Theorem 3.8])**.** *Let $S = K[x_1, \dots, x_n]$. Then every finitely generated graded S-module has a finite graded resolution of length at most n.*

1.3 Tensor Algebra

Let A be a commutative ring and M an A-module. For every integer $n \geq 0$, the A-module *n-th tensor power of M* is denoted by $T^n(M)$ or $M^{\otimes n}$, where $T^0(M) = M^{\otimes 0} = A$. Sum of the tensors powers, forms a graded A-module:

$$\bigotimes M = \bigoplus_{j=0}^{\infty} M^{\otimes j}.$$

We shall define a graded A-algebra structure on $\bigotimes M$. By the assignment:

$$((x_1, \dots, x_m), (y_1, \dots, y_n)) \longmapsto x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n$$

we get an A-bilinear map $M^{\otimes m} \times M^{\otimes n} \to M^{\otimes (m+n)}$. Its additive extension to $\bigotimes M \times \bigotimes M$, gives $\bigotimes M$ the structure of a graded A-algebra with identity 1_A.

The A-algebra $\bigotimes M$ is called *tensor algebra of M*. Obviously, $\bigotimes M$ is not commutative in general. We identify M and $T^1(M)$. The injection $\varphi\colon M \to \bigotimes M$ is called the *canonical injection* of M into $\bigotimes M$. The tensor algebra is characterized by a universal property.

Proposition 1.3.1 (Universal property of tensor algebras, [Bou, Chapter III, §5, Proposition 1])**.** *Let E be an A-algebra and $f\colon M \to E$ an A-linear mapping. Then, there exists unique A-algebra homomorphism $g\colon \bigotimes M \to E$ such that, $f = g \circ \varphi$:*