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A New Method for Testing Interaction in Unreplicated Two-Way Analysis of Variance

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To my family, specially my mother and father
and also to whom it may study

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ABSTRACT

A NEW METHOD FOR TESTING INTEARCTION IN UNREPLICATED TWO-WAY ANAYLSIS OF VARAINCE

BY

MAHMOOD KHARRATI-KOPAEI

For a two-way ANOVA table, with a single observation per cell, the standard approach is to assume that the interaction between the two factors is negligible, and to base inferences about the main factors on the model without interaction. However, there is no totally satisfactory method for testing if interaction can be ignored.

The classical approach is to specify a functional form for the interaction terms, involving a small number of parameters, and then use an appropriate test. But, such tests have low power if the functional form is inappropriate. This has led researchers to propose tests which do not assume a specific form for the interactions.

In this thesis, we present a new approach for testing interaction without assuming a specific form for the interaction. This approach is fairly simple and flexible, and its usefulness is illustrated with several examples. We also present a general result which shows that there is no test of interaction with good power properties against all types of interaction.

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CHAPTER 1

INTRODUCTION

Two-way layout is one of the most widely used experimental designs. Situations for which the two-way designs are appropriate are numerous; for example, in Biology, Medicine, Social Sciences, etc. When the experiment is expensive for decreasing the number of runs, it is usual to apply unreplicated two-way analysis of variance (ANOVA); that is, only one observation is taken per each combination of the two factors.

The general model for an **unreplicated** two-way layout, with two factors A and B, is

$$y_{ij} = \mu_{ij} + \varepsilon_{ij} \tag{1.1.1}$$

($i=1,\ldots,r$; $j=1,\ldots,c$) where y_{ij} denote the observations, μ_{ij} denote the cell means, and ε_{ij} denote random errors which are independent normal random variables with mean zero and variance σ^2.

When the two factors do not have independent effects, i.e. when the effect of one factor depends on the levels of the other factor, they are said to interact (or there is interaction between the two factors). More precisely, there is **interaction** if the difference between any two levels of B is not the same at all levels of A; that is if

$$\mu_{ij} - \mu_{ij'} \neq \mu_{i'j} - \mu_{i'j'}$$

for some $i \neq i'$ and $j \neq j'$. For example, consider Tables 1.1.1 and 1.1.2, which are simple artificial examples with $r = 2$ and $c = 2$.

Table 1.1.1 An example with interaction

Factor B

		Level 1	**Level 2**
Factor A	**Level 1**	$\mu_{11} = 25$	$\mu_{12} = 5$
	Level 2	$\mu_{21} = 5$	$\mu_{22} = 25$

Table 1.1.2. An example without interaction

Factor B

		Level 1	**Level 2**
Factor A	**Level 1**	$\mu_{11} = 20$	$\mu_{12} = 15$
	Level 2	$\mu_{21} = 10$	$\mu_{22} = 5$

In Table 1.1.1, the difference in means between the levels of factor B is not the same at levels 1 and 2 of factor A; i.e. $\mu_{11} - \mu_{12} \neq \mu_{21} - \mu_{22}$; hence there is interaction between the two factors. But in Table 1.1.2, there is no interaction since $\mu_{11} - \mu_{12} = \mu_{21} - \mu_{22} = 5$.

The presence or absence of interaction can be illustrated graphically using the **interaction plot**. It displays the average at each of the combinations of the two factors, using the levels of factor B (or A) as the horizontal axis and the average as the vertical axis (the averages at the same level of A (or B) are joined by a line). For example, Figures 1.1.1 and 1.1.2 are the interaction plots for the data of Tables 1.1.1 and 1.1.2. Graphically, in the absence of interaction, the lines are parallel (see Figure 1.1.2). But if there is interaction, then the lines would not be parallel (see Figure 1.1.1).

Figure 1.1.1 Interaction plot for Table 1.1.1

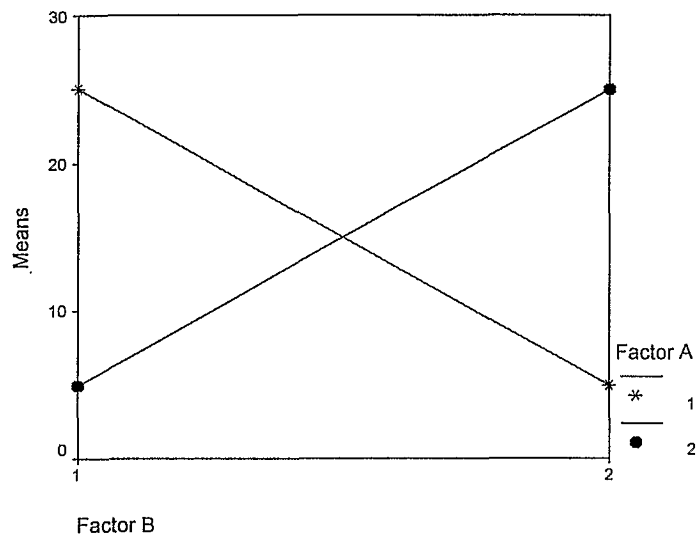
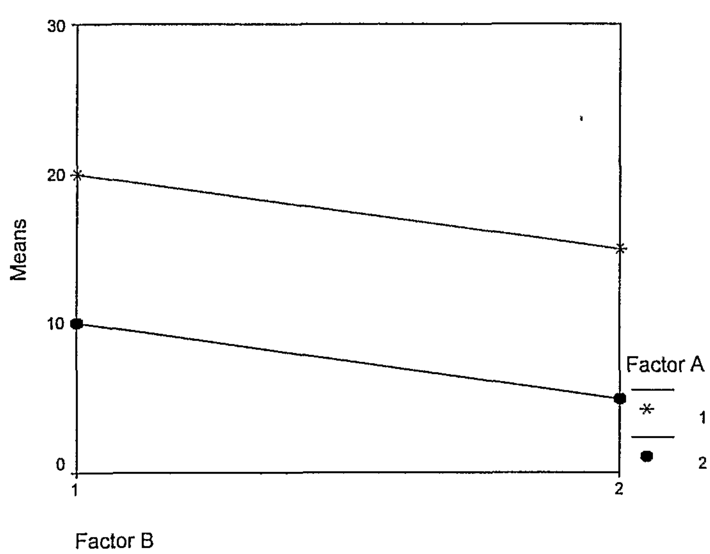


Figure 1.1.2 Interaction plot for Table 1.1.2



When interaction is present, care should be taken when making inferences about the effects of the two factors, since "a significant interaction will often **mask** the significance of main effects" (Montgomery, 1997 p.174). For example, in Table 1.1.1, the two levels of the factor B have the same average ($\overline{\mu}_{.1} = \overline{\mu}_{.2} = (25+5)/2 = 15$). It would be wrong, however, to conclude that factor B has no effect (i.e., to conclude that there is no difference between the levels of factor B): Factor B has an effect, but its effect depends on the level of factor A; it has decreasing effect at level 1 of factor A, but increasing effect at level 2 of factor A. Therefore, it is **necessary** to determine if interaction is present before making inferences about the main effects.

The **main problem** in unreplicated two-way ANOVA model is that all the observations are used to estimate μ_{ij} (each y_{ij} is used to estimate μ_{ij}) and no observation is left to estimate σ^2. Hence the usual F test for testing interaction can not be used (the residual sum of squares of the model 1.1.1 and its degree of freedom are both zero). Therefore, the analysis of unreplicated two-way ANOVA models presents a challenge.

It is convenient to rewrite the cell-means model (1.1.1) in the more usual form

$$y_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ij} \qquad (1.2.1)$$

($i = 1,2,\ldots,r$; $j = 1,2,\ldots,c$; $\sum_{i=1}^{r}\alpha_i = \sum_{j=1}^{c}\beta_j = \sum_{i=1}^{r}\gamma_{ij} = \sum_{j=1}^{c}\gamma_{ij} = 0$), where μ, α_i, β_j, γ_{ij} denote the grand mean, the effects of factor A, the effects of factor B, and the interaction effects, respectively.

For a two-way ANOVA table, with a single observation per cell, the standard approach is to assume that the interaction between the two factors is negligible and to base inferences about the main factors effects on the model without interaction. But there is no totally satisfactory procedure for testing whether the interactions terms can be ignored. One approach is to assume a specific **functional form** for the interaction terms, involving a small number of parameters, and then proceed with an appropriate test. The classical procedure is Tukey's (1949) "one degree of freedom" procedure which assumes that the interactions are of the form $\gamma_{ij} = \lambda\alpha_i\beta_j$ and then proceeds to test if $\lambda = 0$. Other tests, based on more complicated forms for the interaction terms, are proposed by

Mandel (1961, 1971), Johnson and Graybill (1972) and Corsten and Van Eijnsbergen (1972, 1974); see also Hegemann and Johnson (1976) and Yochmowitz and Cornell (1978). As noted by Boik (1993a), the main problem with this approach is that if the functional form for the interactions is **misspecified**, then the power of the test is **low**. It is therefore useful to have a test which does not rely on a specific form for the interactions.

Milliken and Rasmuson (1977) proposed a test based on the sample variance for each row

$$S_i^2 = \frac{1}{c-1}\sum_{j=1}^{c}(y_{ij} - \bar{y}_{i.}),$$

where $\bar{y}_{i.}$ denotes average of y_{ij} over index j. Since $E(S_i^2) = \sigma^2 + \frac{1}{c-1}\sum_{j=1}^{c}(\beta_j + \gamma_{ij})^2$, it follows that if $\gamma_{ij} = 0$, then $E(S_i^2)$ has the same value in all rows; hence any test for checking equality of variances can be used. Their test was later discussed and modified by Piepho (1994). Tusell (1990) proposed a test of interaction which used the likelihood ratio test of sphericity as a test of additivity. Boik (1993a) derived the locally best invariant (under transformations of scalar multiplication and orthogonal rotation of the residual matrix) test by using the residual matrix; see also Boik (1993b). Speed and Speed (1994) proposed an ad-hoc graphical approach.

The organization of this thesis is as follows:

In Chapter 2, we review the most important approaches for testing interaction in unreplicated two-way ANOVA (both procedures: those have been assumed a functional form for the interaction terms and those have not been assumed).

In Chapter 3, we present a new approach for testing interaction which does not assume a special form for the interactions. This approach is based on considering sub-tables of the original $r \times c$ table and doing a simple F test. This method is fairly simple and flexible and its usefulness is illustrated with several examples.

In Chapter 4, we compare the proposed approach with other tests of interaction. We also present a general result regarding the "overall" power of tests of interaction. This result implies that there is no test of interaction with "high" power against all possible alternatives. It also shows that, in the absence of any prior information concerning the pattern of interaction, no test can be preferred to any other test on the basis of "power".

In Chapter 5, we present our conclusions and recommendations.

The Appendix contains the proofs of theorems, the tables of critical values, the computer programs that have been used, and the power simulation results.

CHAPTER 2

REVIEW OF THE LITERATURE

2-1 Introduction

Consider model (1.2.1) for unreplicated two-way layout with two factors A and B in which two factors are applied in r and c levels, respectively. The methods for testing interaction can be categorized into two groups. One group assumes a functional form for the interaction terms and then proceed with an appropriate test; such as, Tukey (1949), Mandel (1961, 71), Johnson and Graybill (1972), Corsten and Van Eijnsbergen (1972,74), Hegmann and Johnson (1976a, b), Yochmowitz and Cornel (1978) and Marasinghe and Johnson (1981, 82). The other group does not assume a special form for the interactions; for example, Milliken and Rasmuson (1977), Tusell (1990), Boik (1993a, b), Piepho (1994), and Speed and Speed (1994). The research of the first group will be considered in Section 2.2, and the second group will be considered in Section 2.3.

2-2 Methods based on a functional form for interactions

Some researchers have proposed tests which assume that the interactions have a specific functional form (involving a small number of parameters). The most important methods are those of Tukey (1949), Mandel (1961, 1971), and Johnson and Graybill (1972). We will discuss these methods in this section.

2-2-1 Tukey's method (1949)

The classical test of interaction is Tukey's (1949) "one degree of freedom" test, which assumes that the interactions are of the form $\gamma_{ij} = \lambda\alpha_i\beta_j$, where λ is a free parameter. In this method, the data are additive (there is no interaction) if $\lambda = 0$. Tukey's test rejects the hypothesis $\lambda = 0$, at level α, when

$$\frac{(rc-r-c)SS_\lambda}{SSE-SS_\lambda} > F_\alpha(1,(r-1)(c-1))$$

where

$$SS_\lambda = \frac{\left(\sum_{i=1}^{r}\sum_{j=1}^{c}(\bar{y}_{i.}-\bar{y}_{..})(\bar{y}_{.j}-\bar{y}_{..})y_{ij}\right)^2}{\sum_{i=1}^{r}(\bar{y}_{i.}-\bar{y}_{..})^2\sum_{j=1}^{c}(\bar{y}_{.j}-\bar{y}_{..})^2},\ SSE = \sum_{i=1}^{r}\sum_{j=1}^{c}(y_{ij}-\bar{y}_{i.}-\bar{y}_{.j}+\bar{y}_{..})^2,$$

and $F_\alpha(1,(r-1)(c-1)-1)$ is the $(1-\alpha)$-th quantile of the F-distribution with 1 and $(r-1)(c-1)-1$ degrees of freedom. (The "dot" notation indicates average over that index.) For more details about Tukey's test, see Wang and Chow (1994, p. 404).

Tukey's test is available in SPSS 11.5 (follow the root: Analyze/ Scale/ Reliability Analysis/ Statistics) and in S-Plus (2000) (see page 523 of the online manual).

2-2-2 Mandel's method (1961)

Mandel (1961) extended Tukey's procedure by allowing γ_{ij} to depend on the effect of one of the factors (but not both). More specifically, he assumed the functional form

$$\gamma_{ij} = \alpha_i \xi_j \text{ or } \gamma_{ij} = \delta_i \beta_j ,$$

where ξ_j and δ_i are parameters subject to $\sum_{i=1}^{r} \delta_i = \sum_{j=1}^{c} \xi_j = 0$. According to the Mandel's model ($\gamma_{ij} = \delta_i \beta_j$), we can write

$$\begin{aligned} y_{ij} &= \mu + \alpha_i + \beta_j + \delta_i \beta_j + \varepsilon_{ij} \\ &= \mu + \alpha_i + \beta_j (1 + \delta_i) + \varepsilon_{ij} \\ &= \alpha_i^* + \beta_j Q_i + \varepsilon_{ij} . \end{aligned}$$

Thus, the data represent r straight lines, corresponding to the r rows, and differing from each other in both slopes and intercepts. If all lines have the same slope, then the data are additive. Based on the above idea, Mandel proposed the following statistic

$$M = \frac{(c-2)\sum_{i=1}^{r}(b_i - 1)^2 \sum_{j=1}^{c} C_j^2}{\sum_{i=1}^{r}\sum_{j=1}^{c}(y_{ij} - \bar{y}_{i.} - b_i C_j)^2},$$

where $b_i = \frac{\sum_{j=1}^{c} y_{ij} C_j}{\sum_{j=1}^{c} C_j^2}$ and $C_j = \bar{y}_{.j} - \bar{y}_{..}$. Additivity is rejected if

$$M > F_{\alpha}(r-1, (r-1)(c-2)) .$$

2-2-3 Mandel's method (1971)

A more general model in which non-additivity does not need to depend on the main effects was proposed by Mandel (1971). This model is based on the singular value decomposition of the interaction matrix $[\gamma_{ij}]_{rxc}$, which can be written as

$$\gamma_{ij} = \sum_{k=1}^{a} \delta_{ik}\xi_{jk}\sqrt{\lambda_k} ,$$

where a is the rank of $[\gamma_{ij}]_{rxc}$; $\sum_{i=1}^{r}\delta_{ik} = \sum_{j=1}^{c}\xi_{jk} = 0$; $[\delta_{ik}]$ and $[\xi_{jk}]$ are orthogonal matrices of order $r \times a$ and $c \times a$ respectively; and λ_k's are the nonzero eigenvalues of $[\gamma_{ij}]'[\gamma_{ij}]$ (A' denotes the transpose of the matrix A). The rank of the interaction matrix is not known in practice, and if the data are additive, then the rank of this matrix will be zero. Therefore Mandel introduced the hypothesis $H_0 : a = 0$ versus $H_t : a = a_t$, where a_t is a positive integer number.

2-2-4 Johnson and Graybill's method (1972)

Johnson and Graybill (1972) used Mandel's (1971) model and derived the likelihood ratio test of $H_0 : a = 0$ versus $H_1 : a = 1$. They showed that H_0 is rejected when

$$JG = \frac{l_1}{l_1 + l_2 + \cdots + l_{c-1}} > k,$$

for some k, where $l_1 \geq l_2 \geq \ldots \geq l_{c-1}$ are the non-zero eigenvalues of $Z'Z$ in which $Z = [y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..}]_{rxc}$.