In the name of GOD

SM supermanifolds can be given the structure of SSM supermanifolds.

A DISSERTATION

Submitted as a partial requirement for fulfillment of the degree of master of Science to the :

Department of Mathematics

of

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" SM supermanifolds can be given the structiure of SSM super manifolds "

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ABSTRACT: SM and SSM functions differ by constant functions as defiend by Batchelor[2], She has shown that any SSM supermanifold can be given the structure of an SM supermanifold.

Following her ideas, we prove that the composition of SM functions is an s SM function, more over, we show that an SM supermanifold, can be given the structure of an SSM super manifold .

The definitions and terminology used in[2]are employed throughout the paper.

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Introduction:

1-0 STANDARD NOTATION

All work is done over the real numbers R, but can be generalized to the complex numbers C.

1-1 Modules

Let R be a ring. A(left)R-module is an additive abelian group A together with a function.

$R \times A \longrightarrow A$ such that for all $r,s \in R$

$(r,a) \longrightarrow ra$

and for any $a,b \in A$:

i) $r(a+b) = ra+rb$.

ii) $(r+s)a = ra+sa$.

iii) $r(sa) = (rs)a$.

If R has an identity element 1_R and

iv) $1_R a = a$ for all $a \in A$.

Theh A is said to be a unitary R-module .

If R is a division ring , then a unitary R-module is called a (left) Vector Space.

A(unitary) right R-module is defined similarly via a function

$A \times R \longrightarrow A$.

$(a,r) \longrightarrow ar$

Where $a \in R$, v, v_1, v_2 belong to V and $w, w_1, w_2 \in W$.

1-3 Some Properties of the Tensor Product

a) Universal Mapping Property.

Let φ denote the bilinear map $(v,w) \longrightarrow v \otimes w$ of $V \times W$ into $V \otimes W$. Then whenever U is a vector space and $l: V \times W \longrightarrow U$ is a bilinear map, there exists a unique linear map $\tilde{l}: V \otimes W \longrightarrow U$ such that the following diagram commutes:

$$\begin{tikzcd} V \otimes W \arrow[dr, "\tilde{l}"] & \\ V \times W \arrow[u, "\varphi"] \arrow[r, "l"'] & U \end{tikzcd}$$

The pair consisting of $V \otimes W$ and φ is said to solve the universal mapping problem for bilinear maps with domain $V \times W$. $V \otimes W$ and φ are unique with this property.

b) $V \otimes W$ is canonically isomorphic with $W \otimes V$.

c) $V \otimes (W \otimes U)$ is canonically isomorphic with $(V \otimes W) \otimes U$.

d) Let $\{e_i : i = 1, \ldots, c\}$ and $\{f_j : j = 1, \ldots, d\}$ be bases for V and W respectively, then $\{e_i \otimes f_j ;\ i = 1, \ldots, c \text{ and } j = 1, \ldots, d\}$ is a basis of $V \otimes W$.

1-4 Topological Spaces

A Topological Space is a set X in which a collection of subsets(called opensets) has been specified, with the following properties: X is open, $\emptyset$ is open , the intersection of any two open sets is open,and the union of every collection of open sets is open. Such a collection is called a Topology on X.

A set $E \subset X$ is closed if and only if its complement is open. A neighborhood of a point $p \in X$ is any open set that contains P.

(X,τ) is a Hausdorff space, and τ is a Hausdorff topology, if distinct points of X have disjoint neighborhoods. A collection $\tau' \subset \tau$ is a base for τ if every member of τ (that is, every open set) is a union of members of τ'.

A collection γ of neighborhoods of a point $p \in X$ is a local base at p If every neighborhood of P contains a member of γ.

1-5 Normed Spaces

A vector space X is said to be a normed space if to every $x \in X$ there is associated a non-negative real number $\|x\|$, called the norm of x, in such a way that:

a) $\|x + y\| \leq \|x\| + \|y\|$ for all x and y in X;

b) $\|\alpha x\| = |\alpha| \, \|x\|$ if $x \in X$ and α is a scalar,

c) $\|x\| > 0$ if $x \neq 0$

Every normed space may be regarded as a metric space, in which the distance d(x,y) between x and y is $\|x-y\|$.

The relevant properties of d are:

i) $0 \leq d(x,y) < \infty$ for all x and y,

ii) $d(x,y) = 0$ if and only if $x = y$,

iii) $d(x,y) = d(y,x)$ for all x and y,

iv) $d(x,z) \leq d(x,y) + d(y,z)$ for all x,y,z.

1-6 Banach Space: A Banach space is a normed space which is complete in the metric defined by its norm; This means that every cauchy sequence is required to be convergent.

1-7 Definition: A set $C \subseteq X$ is said to be convex if

$$tC + (1-t)C \subseteq C \qquad (0 \leqslant t \leqslant 1).$$

In other words, it is required that C should contain $tx+(1-t)y$ if $x \in C, y \in C$, and $0 \leqslant t \leqslant 1$.

1-8 Definition: Suppose τ is a topology on a vector space X such that:

a) Every point of X is a closed set.

b) The vector space operations are continuous with respect to τ.

Under these conditions, τ is said to be a vector Topology on X, and X is a Topological vector space.

Definition : A metric d on a vector space X is called invariant if

$$d(x + z,\ y + z) = d(x,y) \quad \text{for all } x,y,z \text{ in } X.$$

1-9 Definitions: Let X be a topological vector space, with Topology τ.

a) X is locally Convex if there is a local base β whose members are convex.

b) X is an F-space if its Topology τ is induced by a complete invariant metric d.

c) X is a Fréchet space if X is a locally convex F-space.

1-10 Definition: A seminorm on a vector space X is a real - valued function P on X such that

a) $P(x+y) \leqslant P(x)+P(y)$

b) $P(\alpha x) = |\alpha| P(x)$

for all x and y in X and all scalars α.

1-11 Definition: Let X be a Topological vector space, with Topology $\mathcal{T}$. We say X has the Heine - Borel property if every closed and bounded subset of X is compact.

1-12 Notation : Let $\alpha=(\alpha_1,\dots,\alpha_n)$ of non-negative integers α_i and

$$D^{\alpha}=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

Where order α is; $|\alpha|=\alpha_1+\dots+\alpha_n$. If $|\alpha|=0$, $D^{\alpha}f=f$. α is called multi-index.

2-0 The Space $C^{\infty}(\Omega)$

Suppose Ω is a nonempty open set in some euclidean space. We now define a Topology on $C^{\infty}(\Omega)$ which makes $C^{\infty}(\Omega)$ into a Fréchet space with the Heine-Borel property:

To do this, choose compact sets K_i, $i=1,2,3,\dots$ such that K_i lies in the interior of K_{i+1} and $\Omega=\cup K_i$.

Define seminorms P_N on $C^{\infty}(\Omega)$, $N=1,2,3,\dots$ by setting

$$P_N(f)=\max\left\{\left|D^{\alpha}f(x)\right| : x\in K_N,\ |\alpha|\leq N\right\}.$$

A local base is given by the sets

$$V_N=\left\{f\in C^{\infty}(\Omega) : P_N(f)<\frac{1}{N}\right\} \qquad (N=1,2,3,\dots)$$

If $\{f_i\}$ is a cauchy sequence in $C^{\infty}(\Omega)$, and if N is fixed, then $f_i-f_j\in V_N$ if i and j are sufficiently large. Thus $\left|D^{\alpha}f_i-D^{\alpha}_jf\right|<\frac{1}{N}$ on K_N, if $|\alpha|\leq N$. It follows that each $D^{\alpha}f_i$ Converges (uniformly on compact subsets of Ω) to a function g_{α}.

In particular, $f_i(x)\longrightarrow g_0(x)$. It is now evident that $g_0\in C^{\infty}(\Omega)$, that $g_{\alpha}=D^{\alpha}g_0$, and that $f_i\longrightarrow g$ in the topology of $C^{\infty}(\Omega)$. Thus $C^{\infty}(\Omega)$ is a Fréchet space.

3-1 Algebra

Let K be a commutative ring with identity.

A K- algebra (or algebra over K) A is a ring A such that:

i) $(A,+)$ is a unitary (left)K-module;

ii) $K(ab)=(ka)b=\alpha(kb)$ for all $\overset{k}{K} \in K$ and $a,b \in A$,

A K- algebra A which , as a ring ,is a division ring ,is called a

<u>division algebra</u>.

An algebra over K may be defined as Triple(A,M,U) with A a k-space, $M:A\otimes A \longrightarrow A$ a map called multiplication, $U:K \longrightarrow A$ a map called the Unit map,and such that the following two diagrams are commutative:

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "I\otimes M"] \arrow[d, "M\otimes I"'] & A\otimes A \arrow[d, "M"] \\
A\otimes A \arrow[r, "M"'] & A
\end{tikzcd}$$

(Associativity of M)

$$\begin{tikzcd}
K\otimes A \arrow[r, "U\otimes I"] \arrow[dr] & A\otimes A \arrow[d] & A\otimes K \arrow[l, "I\otimes U"'] \arrow[dl] \\
 & A &
\end{tikzcd}$$

(Unitary property).

In the second diagram the maps $A\otimes K \longrightarrow A$ and $K\otimes A \longrightarrow A$ are the natural isomorphisms.

Here I is identity map and $U(1_k)$ corresponds to the identity of A, where 1_k is the identity of K .

3-2 Grading

A vector space v over a field K(R or C) is a (Z_2 -graded) graded vector space if one has fixed subspace V_0 and V_1, called, the even and odd parts of V respectively and $V = V_0 \oplus V_1$.

The subspaces V_0 and V_1 are also referred to as the homogeneous components of V .

An elemnt $v \in V$ is called even if $v \in V_0$ and odd if $v \in V_1$. It is called homogeneous if V is either even or odd. If $V = V_0 + V_1 \in V$ where $V_i \in V_i$, then V_0 is the even componet and V_1 the odd componet of V, and the elements V_0, V_1 are called the homogeneous componets of V.

Also if $v \in V_i$ then i is called the degree of the homogeneous element v and if $V \neq 0$ we put $|v| = i$.

A subspace $W \subseteq V$ is called graded if :

$$W = W \cap V_0 + W \cap V_1$$

If V,W are graded vector spaces so is

$$V \oplus W \quad \text{where } (v+W)_i = v_i \oplus W_i \quad \text{and}$$

$$V \otimes W \quad \text{where } (v \otimes w)_i = \sum_{j+k=i} v_j \otimes w_k$$

Also $\mathrm{Hom}_k(v,w)$ is graded where $\alpha \in \mathrm{Hom}_k(v,w)$ is homogeneous of degree $|\alpha|$ if $\alpha(v_i) \subseteq W_{i+|\alpha|(\mathrm{mod}\,2)}$ for $i = 0,1$.

In particular End V is graded; $\mathrm{End}(v) = (\mathrm{End}(V))_0 + (\mathrm{End}(v))_1$ where

$$(\mathrm{End}(v))_i = \{\alpha \in \mathrm{End}(v) : \alpha V_j \subset V_{i+j(\mathrm{mod}\,2)}\} .$$

3-3 Graded Algebra

Let B be an algebra with a unity ,i.e, $1 \in B$.

An algebra B is called a graded algebra if B is a graded vector space such that $B_i B_j \subseteq B_{i+j}$ and $1 \in B_0$, in orther words $|ab| = |a| + |b|$ for $a,b \in B$.

Two elements x,y in a graded algebra B are graded commutitive if any homogeneous componet of x is graded commutitive with any homogeneous componet of y .

If x and y are homogeneous then they are graded commutative if either one is zero or other wise $xy = (-1)^{|x||y|} yx$.

A graded algebra B is graded commutative (Algebra) if one has $xy=(-1)^{|x||y|}yx$ for $x,y \in B$.

By a left module V for the graded algebra B we mean that V is a left module in the usual sense but that V is also a graded vector space and $B_i V_j \subseteq V_{i+j}$. V is called left graded B- Module.

If V is a left module for a graded commtative algebra B, then inherits a right module struture where we define:

$$V.b = (-1)^{|b||v|} b.v$$

3-4 Let

$T_B^n(v) = B \otimes v \otimes \ldots \otimes v$, define $T_B(v) = \bigoplus_{n=0}^{\infty} T_B^n(v)$.

$T_B(v)$ is Z-graded and also Z_2 - graded, therefore $T_B(v)$ is a bigraded $(z + z_2)$ algebra.

The <u>graded symmetric algebra</u> $S_B(v)$ is $T_B(v)/_I$, where I is ideal in T_B (v) generated by all elements in $T_B(v)$ of the form $x\otimes y-(-1)^{|x|\,|y|}y\otimes x$ for $x,y\in V$,

i.e, $I=\langle v\ \ T_B(v):V=x\otimes y-(-1)^{|x|\,|y|}y\otimes x\rangle$.

<u>The exterior algebra $\Lambda_B(v)$</u> of V over B is quotient $T_B(v)/_J$ where J is ideal in T_B (v) generated by all elements in T_B (v) of the form $x\otimes y+(-1)^{|x|\,|y|}y\otimes x$

for $x,y\in V$.

$S_B(v)$ and $\Lambda_B(v)$ are bigraded ($Z\oplus Z_2$) algebras and $\underset{B}{\Lambda}(V)$ is a graded commutative algebra.

In the case $B=K$ we drop the subscript B .Thus if V is vector space and

$(\Lambda V)_K=\{V_{i_1}\wedge\cdots\wedge V_{i_k}:1\leqslant i_1\leqslant i_2<\cdots<i\}$,where $\{v_1,v_2,\ldots\}$ form a basis for v,then $V=\underset{k}{\oplus}(\Lambda V)_k$, and $V\subset(\Lambda V)_1$

<u>3-5</u> <u>Example</u> :Let $B=\Lambda R^L$ for $L<\infty$ .Thus B is generated by L generators $\gamma_1,\ldots,\gamma_L$ .There is an obvious r basis for with $B_0=\langle\gamma_{i_1}\wedge\cdots\wedge\gamma_{i_k}:$ $1\leqslant i_1<\cdots<i_k\leqslant L$, keven$\rangle$ and $B_1=\langle\gamma_{i_1}\wedge\cdots\wedge\gamma_{i_k}:1\leqslant i_1<\cdots<i_k\leqslant L,\text{kodd}\rangle$.

Multiplication in ΛR^L is given by justaposition :

$$(\gamma_{i_1}\wedge\cdots\wedge\gamma_{i_k})(\gamma_{i'_1}\wedge\cdots\wedge\gamma_{i'_{k'}})=\gamma_{i_1}\wedge\cdots\wedge\gamma_{i_k}\wedge\gamma_{i'_1}\wedge\cdots\wedge\gamma_{i'_{k'}} .$$

One can define a symmetric bilinear form on B by declaring elements of the above basis to be orthonormal.

This gives B the structure of a Banach algebra,.

<u>3-6</u> <u>The Augemetation map E</u>

For any of the graded commutative algebras B defind above there is a unique algebra homomor-phism $\varepsilon:B\longrightarrow R$ given by sending the odd generators to zero.

3-7 Coarse Topology

This Topology is defind by carrying back the Topology on R via The augmentation map.

Say a set $U \subset B$ is in the coarse Topology on B if $U = \epsilon^{-1}(V)$ for some open set V in R .

3-8 Supereuclidean Space

Given a Z_2 graded commutative algebra B , define (r,s) dimensional supereuclidean space $E_B^{r,s} = (B_0)^r \times (B_1)^s$.

If B has a Topology , B_0 and B_1 have Topologies as subspaces of B, and $E_B^{r,s}$ has the Topology of a product of Topological spaces.

Using the coarse Topology for $B = \Lambda R^L$, This means that a set $U \subset E_B^{r,s}$ is open if and only if $U = \epsilon^{-1}(V)$, where V is an open set in R and $\epsilon : E_B^{r,s} \longrightarrow R^r$ is given by $\epsilon(u_1,\ldots,u_r,v_1,\ldots,v_s) = (\epsilon(v_1),\ldots,\epsilon(u_r))$.

Generally one writes $E^{r,s}$ for $E_B^{r,s}$.

4-0 SSM smooth Maps and SM smooth Maps

4-1 Let

$B = \Lambda R^L$, $L < \infty$, with the coarse topology .

Let U be a (coarse) open set in $E^{r,s}$ and let F(U) denote the algebra of all functions from U to B .

For SSM smooth maps the following subalgebras of F(U) are considered smooth:

i) The algebra P generated over the real numbers by the coordinate projections.

In other words : in $F(U)$ there are projections p_i , $i = 1,\ldots,r$ and Π_j, $j = 1,\ldots,s$ given by $p_i\ (u_1,\ldots,u_r\ ,v_1,\ldots,v_s\ ) = u_i$,

$$\Pi_j(u_1,\ldots,u_r,v_1,\ldots,v_s\ )= v_j\ .$$

Define the algebra of <u>superpolynomials P</u> to be the (real) sabalgebra of $F(U)$ generated by the projections.

We denate the symmetric algebra generated by the projections p_i , by $sym\ (p_i)$, and let $\Lambda\ (\Pi_j)$ denate the exterior algebra generated by the elements Π_j .

Let $j: P \longrightarrow F\ (U)$ denate the inclusion, For example if $p_1^{i_1} p_2^{i_2},\ldots p_r^{i_r}\ldots\Pi_s^{j_s}$ beloggs to P and suppose $(u_1,\ldots,u_r,V_1,\ldots,V_s) \in U \subset E^{r,s}$, then

$$J(p_1^{i1}\ldots \Pi^{j1}\ldots\Pi_s^{js})(U_1,\ldots,U_r,v_1,\ldots,v_s) = U_1^{i_1}\wedge\ldots\ U_r^{i_r}\wedge V_1^{j_1}\wedge\ \ldots\ \wedge V_s^{js}$$

ii) The algebra $C^\infty\ (\mathcal{E}\ (U))$ included in $F(U)$ in the following way:

Let $x_1,\ldots,x_r$ be a set of coorinates for R^r.

If $\mu=(i_1,\ldots,i_r)$ is an r-tuple of non-negative integers and let $\partial(\mu)$ be the differential opreator on R^r given by $\partial(\mu) = \dfrac{\partial^{\sum i_j}}{\partial x_1^{i_1}\cdots\partial x_r^{i_r}}$ and let $P(\mu)(V_1,\ldots,V_r)= U_1^{i_1}\wedge\ldots\ \wedge U_r^{i_r}$ and $a(\mu)$ be the constant.

Define $K:C^\infty\ (\mathcal{E}\ (U)) \longrightarrow F(U)$ by

$$Kf(U_1,\ldots U_r,V_1,\ldots,V_s)= \sum_\mu a(\mu)\,\partial(\mu)f(\mathcal{E}(U_1),\ldots,\mathcal{E}(U_r))P(\mu)((U_1-\mathcal{E}(U_1)), \ldots,(U_r-\mathcal{E}(U_r)))\ .$$

K is an algebra homorphism.

4-2 Definiton:

The set of SSM Smoth maps is the image of the map $J\otimes K:P\otimes C^{\infty}(\mathcal{E}(U))\longrightarrow F(U)$.

A function $f:U\longrightarrow E^{r',s'}$ is SSM if f composed with projection onto any coordinate projection is SSM.

4-3 :

We add the following to the list of " approved smooth functions" in 4-1

iii) The constant functions.

Write $C:B\longrightarrow F(U)$ for the inclusion of B in F(U) .

4-4 Definition

The SM Smooth functions on a coarse open set $U\subset E^{r,s}$ are those in the image of the map: $C\otimes J\otimes K:B\otimes P\otimes C^{\infty}(\mathcal{E}(U))\longrightarrow F(U)$.

A function $f:u\longrightarrow E^{r',s'}$ is SM if f composed with projection onto any coordinate projection is SM.

4-5 Remark:

We define an algebra SP to be the algebra of superpolynomials with coefficients in $B=\Lambda R^{L}$, that is the algebras over B generated by the projections P_i and .
Thus $\Lambda R^{L}\otimes P=SP$, and $SM(U)\cong B\otimes SSM(U)$.

4-6 Proposition :

Let U be a coarse open set in E^{r_1,s_1} and V is a coarse open set in $E^{r2,s2}$.
If f is in SM(U,V) and g is in $SM(V,E^{r_3,s_3})$, then g of is in $SM(U,E^{r_3,s_3})$.

Proof:

Case 1; Let $f\in SM(U,V)$ and $g\in SM(V,\Lambda R^{L})$,where g is coordinate projection, then g of is in $SM(U,\Lambda R^{L})$, by the definition SM Smooth maps.

Case 2 : If $f \in SM(U,V)$ and $g \in SM(V, \wedge R^L)$, that g is in $SP(V, \wedge R^L)$,

that is, g is an element of algebra of super polynomials with coefficients in $\wedge R^L$,Thus :

$$g = \sum a_{\mu,\nu}\, p_1^{i1} \cdots p_{r2}^{ir_2} \cdots \eta_{s_2}^{\gamma_{s_2}}$$,that $a_{\mu,\nu}$ is in $\wedge R^L$.

Now, we have : $g \circ f = \left(\sum_{\mu,\nu} a_{\mu,\nu}\, p_1^{i1} \cdots p_{r2}^{ir2} \cdots \eta_{s_2}^{\gamma_{s_2}}\right) \circ f =$

$$\sum_{\mu,\nu} a_{\mu,\nu} (p_1^{i1} \circ f) \cdots (p_{r2}^{ir2} \circ f) \cdots (\eta_{s_2}^{\gamma_{s_2}} \circ f)$$

That is, gof is in $SM(U, \wedge R^L)$.

Case 3: Let $f \in SM(U_1, U_2)$ and $g \in SM(U_2, \wedge R^L)$.

First; topologize $\wedge R^L \otimes P \otimes C^\infty(\mathcal{E}(U))$ by product topology of 2^{L+L} ($= 2^{2L}$) copies $C^\infty(\mathcal{E}(U))$ by 2-0 where $\Omega = \mathcal{E}(U)$.

Let $SM(U, \wedge R^L) = \wedge R^L \otimes P \otimes K(C^\infty(\mathcal{E}(U)))$. Since $SM(U, \wedge R^L)$ is homeomorphic with $\wedge R^L \otimes P \otimes C^\infty(\mathcal{E}(U))$, there-fore $SM(U, \wedge R^L)$ can be topologized by topology $\wedge R^L \otimes P \otimes C^\infty(\mathcal{E}(U))$. So , $SM(U, \wedge R^L)$ is complete and $SP(U)$ is dense in $SM(U, \wedge R^L)$. [see-2-0] ,where SP(U) is algebra of super polynomials over $\wedge R^L$.

Also $SM(U_1, U_2)$ has aatopology induced by the inclusion

$SM(U_1, U_2) \longrightarrow (SM(U_1, \wedge R^L))^{r2+s2}$ given by the projections,. The algebra $SP(r_2, S_2)$ of super polynomials on $E^{r2,s2}$ has a topology as a subspace of $SM(U_2, \wedge R^L)$.

Now , we prove the proposition:

We consider the continuous map determined by composition

$$G: SM(U_1, U_2) \times SP(r_2, s_2) \longrightarrow SM(U_1, R^L).$$

$$(f\ ,\ g_i) \longrightarrow g_i \circ f$$