IN THE NAME OF ALLAH

INTERPOLATING BLASCHKE PRODUCTS

BY

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Mathematics Spans All Dimensions

ABSTRACT

INTERPOLATING BLASCHKE PRODUCTS

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As it has shown by Forstman, [5], Blaschke products are norm dense in the set of inner functions. Later, [8], Marshall showed that H^∞ is the closure of linear span of the set of all Blaschke products. After this the same question about interpolating Blaschke products raised in [6] by Garnett. The problem had reminded open for about 15 years and at last answered by Garnett, [7].

In this thesis we will review this story from Fortsman to Garnett and from [5] to [7]!

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Chapter I

Introduction

In this chapter we will review some important theorems and basic definitions that will be used in the next two chapters.

Schwarz's Lemma

Let $\mathcal{B}$ denote the set of analytic functions from D into $\overline{D}$. The simple but surprisingly powerful Schwarz lemma is this:

Lemma 1.1. *Let $f(z)$ be analytic on a disc $B(0, R_1)$ and suppose that $|f(z)| < R_2$ on $B(0, R_1)$ and $f(0) = 0$. Then*

$$|f(z)| \leq \frac{R_2}{R_1}|z| \quad \text{for } |z| < R_1. \tag{1}$$

Strict inequality holds in (1) for every $z \neq 0$ unless f is of the form

$$f(z) = \frac{R_2}{R_1}e^{i\alpha}z \quad \text{for some real } \alpha.$$

Proof. See [6]. □

Lemma 1.2. *Let $f \in \mathcal{B}$, then*

$$\frac{|f(z) - f(z_0)|}{|1 - \overline{f(z_0)}f(z)|} \leq \left|\frac{z - z_0}{1 - \overline{z_0}z}\right|, \; z \neq z_0,$$

and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}. \tag{2}$$

Equality holds at some point z iff $f(z)$ is a **Mobius transformation.** *(A mobius transformation is any function of the form $T(z) = e^{i\theta}((z - z_0)/(1 - \overline{z_0}z))$.)*

Proof. See [6]. □

The **pseudohyperbolic distance** on D is defined by

$$\rho(z,w) = |\frac{z-w}{1-\overline{w}z}|.$$

Lemma 1.2 says that analytic mappings from D to D are Lipschitz continuous in the pseudohyperbolic distance:

$$\rho(f(z), f(w)) \leq \rho(z,w).$$

The lemma also says that the distance $\rho(z,w)$ is invariant under Mobius transformations:

$$\rho(z,w) = \rho(T(z), T(w)).$$

We write $K(z_0, r)$ for the non-Euclidean disc

$$K(z_0, r) = \{z \mid \rho(z, z_0) < r\},\ 0 < r < 1.$$

Since the family $\mathcal{B}$ is invariant under Mobius transformations, the study of the restrictions to $K(z_0,r)$ of functions in $\mathcal{B}$ is the same as the study of their restrictions to $K(0,r) = \{w \mid |w| < r\}$. In such a study, however, we must give $K(z_0,r)$ the conformal coordinate function $w = T(z) = (z-z_0)/(1-\overline{z_0}z)$. For example, the expression

$$(1-|z|^2)|f'(z)| \tag{3}$$

is conformally invariant. The proof of this fact uses the important identity

$$1-|\frac{z-z_0}{1-\overline{z_0}z}|^2 = \frac{(1-|z|^2)(1-|z_0|^2)}{|1-\overline{z_0}z|^2} = (1-|z|^2)|T'(z)|,$$

which is (2) with equality for $f(z) = T(z)$. Hence if $f(z) = g(T(z)) = g(w)$, then

$$|f'(z)|(1-|z|^2) = |g'(w)||T'(z)|(1-|z|^2) = |g'(w)|(1-|w|^2)$$

and this is what is meant by invariance of (3).

The non-Euclidean disc $K(z_0, r), 0 < r < 1$, is the inverse image of the disc $|w| < r$ under

$$w = T(z) = \frac{z - z_0}{1 - \overline{z_0} z}.$$

Consequently $K(z_0, r)$ is also a euclidean disc $B(c, R)$ and as such it has center

$$c = \frac{1 - r^2}{1 - r^2|z_0|^2} z_0$$

and radius

$$R = r\frac{1 - |z_0|^2}{1 - r^2|z_0|^2}.$$

The pseudohyperbolic distance is a metric on D. The triangle inequality for ρ follows from

Lemma 1.3. *For any three points z_1, z_2, z_3 in D,*

$$\frac{\rho(z_0, z_2) - \rho(z_2, z_1)}{1 - \rho(z_0, z_2)\rho(z_2, z_1)} \leq \rho(z_0, z_1) \leq \frac{\rho(z_0, z_2) + \rho(z_2, z_1)}{1 + \rho(z_0, z_2)\rho(z_2, z_1)}.$$

Proof. See [6]. □

Every Mobius transformation $w(z)$ sending z_0 to w_0 can be written as

$$\frac{w - w_0}{1 - \overline{w_0} w} = e^{i\theta} \frac{z - z_0}{1 - \overline{z_0} z}.$$

Differentiation then gives

$$|w'(z_0)| = \frac{1 - |w_0|^2}{1 - |z_0|^2}. \tag{4}$$

This identity we have already encountered as (2) with equality. By (4) the expression

$$ds = \frac{2|dz|}{1 - |z|^2} \tag{5}$$

is a conformal invariant of the disc. We can use (5) to define the hyperbolic length of a rectifiable arc γ in D as

$$\int_\gamma \frac{2|dz|}{1-|z|^2}.$$

We can then define the **Poincare metric** $\psi(z_1, z_2)$ as the infimum of the hyperbolic lengths of the arcs in D joining z_1 to z_2. The distance $\psi(z_1, z_2)$ is then conformally invariant. If $z_1 = 0, z_2 = r > 0$, it is not difficult to see that

$$\psi(z_1, z_2) = 2\int_0^r \frac{dx}{1-|x|^2} = \log\frac{1+r}{1-r}.$$

Since any pair of points z_1 and z_2 can be mapped to 0 and

$$\rho(z_1, z_2) = |\frac{z_2 - z_1}{1-\overline{z_1}z_2}|,$$

respectively, by a Mobius transformation, we therefore have

$$\psi(z_1, z_2) = \log\frac{1+\rho(z_1, z_2)}{1-\rho(z_1, z_2)}.$$

A calculation then gives

$$\rho(z_1, z_2) = \frac{\tanh(\psi(z_1, z_2))}{2}$$

Moreover because the shortest path from 0 to r is the radius, the geodesics, or paths of shortest distance, in the Poincare metric consist of the images the diameter under all Mubius transformations. These are the diameters of D and the circular arcs in D orthogonal to ∂D. If there arcs are called lines, we have a model of hyperbolic geometry of Lobachevsky.

Hyperbolic geometry is somewhat simpler in the upper half plane

$$\mathcal{H} = \{z = x + iy \mid y > 0\}.$$

In $\mathcal{H}$,

$$\rho(z_1, z_2) = |\frac{z_1 - z_2}{z_1 - \overline{z_2}}|$$

and the element of the hyperbolic arc length is

$$ds = \frac{|dz|}{y}$$

Geodesics are vertical lines and circles orthogonal to the real axis. In $\mathcal{H}$ any two squares

$$\{(x,y) \mid x_0 < x < x_0 + h, h < y < 2h\}$$

are congruent in the non-Euclidean geometry. The corresponding congruent figures in D are more complicated. For there and for other reasons, $\mathcal{H}$ if often more convenient domain for many problems.

Harmonic Functions

Given an open and connected plane set Ω and a function $u : \Omega \longrightarrow \mathbf{R}$ we rise the question: Under what conditions is u the real part of an analytic function $f : \Omega \longrightarrow \mathbf{C}$?

For such a function $f = u + iv$ to exist it is necessary that u belong to the class $C^{\infty}(D)$. Furthermore, in the light of the Cauchy-Reimann relations the existence of f impose another significant constraint on u: $u_{xx} + u_{yy} = 0$ throughout Ω. Indeed, we have

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = 0.$$

The partial differential equation $u_{xx} + v_{yy} = 0$ is known as **Laplace's equation.** We use the symbol Δu as an abbreviation for $u_{xx} + u_{yy}$. thus the Laplace equation is frequently written in the form $\Delta u = 0$. The solutions of this equation are called harmonic functions.

Definition 1.1. *Let Ω be a nonempty open subset of* $\mathbf{C}$.

1) Let $f \in L^1_{loc}(\Omega)$ (locally integrable function in Ω). For every closed *disc* $\overline{B}(z,r) \subset \Omega$, *we call area average of* f *over* $\overline{B}(z,r)$, *and denote it* $A(f,z,r)$, *the complex number given by*

$$A(f,z,r) = \frac{1}{\pi r^2} \int_{\overline{B}(z,r)} f dm,$$

where dm *represent the Lebesgue measure in* $\mathbf{C}$.

2) Let $f : \Omega \longrightarrow \mathbf{C}, z \in \Omega, r > 0$ *be such that* $\overline{B}(z,r) \subseteq \Omega$ *and*

$$f|_{\partial B(z,r)} \in L^1.$$

We denote $\lambda(f,z,r)$ *the circular average of* f *over* $\partial B(z,r)$ *by the complex number given by*

$$\lambda(f,z,r) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta.$$

Proposition 1.1. *Let* Ω *be a nonempty open subset of* $\mathbf{C}$ *then the following statements are equivalent for any* $f : \Omega \longrightarrow \mathbf{C}$.

1) $C^2(\Omega)$ *and* $f(z) = \lambda(f,z,r)$ *for every* $\overline{B}(z,r) \subseteq \Omega$.

2) $C^2(\Omega)$ *and* $f(z) = A(f,z,r)$ *for every* $\overline{B}(z,r) \subseteq \Omega$.

3) $f \in L^1_{loc}(\Omega)$ *and* $f(z) = A(f,z,r)$ *for every* $\overline{B}(z,r) \subseteq \Omega$.

Proof. See [1]. □

In the case that f satisfies any of these properties, we say that it is a **Harmonic function** in Ω.

Suppose $u : \Omega \longrightarrow \mathbf{R}$ is harmonic, does there exists a harmonic function $v : \Omega \longrightarrow \mathbf{R}$ such that the function $f = u + iv$ is analytic in Ω? Any function v that fits this description is termed a **Conjugate harmonic function** for u in Ω.

Proposition 1.2. *Let* Ω *be a simply connected open set in the complex plane. Then every real valued harmonic function is* Ω *possess a harmonic conjugate.*

Proof. See [1]. □

Subharmonic Functions

Definition 1.2. *Let X be a topological space. A function $u : X \longrightarrow [-\infty, \infty)$ is said to be upper semi-continuous (usc) if the following hold:*

> *To every $x_0 \in X$ and $M > u(x_0)$ there exists a neighborhood U of x_0 such that $M > u(x)$ for all $x \in U$.*

Remark 1.1. *In the case $X \subseteq \mathbf{C}$ and open, the above definition is equivalent to*

$$u(z_0) \geq \limsup_{z \longrightarrow z_0} u(z) \quad \text{for every } z_0 \in X.$$

By the following three lemmas we review some important properties of usc functions.

Lemma 1.4.

i) If u_1 and u_2 are usc then so are $u_1 + u_2$ and $\max\{u_1, u_2\}$.

ii) If u is usc and $\lambda \in [0, \infty)$ then λu is usc.

iii) If u_α is usc for each α in a non-empty index set A then $\inf_{\alpha \in A} u_\alpha$ is also usc.

Lemma 1.5. *An upper semi-continuous function on a compact topological space attains its supremum. In particular it is bounded above.*

Lemma 1.6. *Let $u : \Omega \longrightarrow [-\infty, \infty)$ where Ω is an open subset of $\mathbf{C}$. Then u is usc iff there exist a decreasing sequence $u_1 \geq u_2 \geq \cdots$ of continuous real valued functions on Ω such that*

$$\lim_{n \longrightarrow \infty} u_n(z) = u(z) \quad \text{for each } z \in \Omega.$$

Proof. The lemma is trivial in case $u(z) = -\infty$ for all z, so we will from now assume that there exists a $z_0 \in \Omega$ such that $u(z_0) \in \mathbf{R}$

The if direction is true by part (iii) of the lemma above. So assume from now on that u is usc.

We will first consider the special case of u being bounded from above, say $u(z) < M$ for all $z \in \Omega$. The trick is here to consider the functions $u_n, n \in \mathbb{N}$ given by

$$u_n(z) = \sup_{y \in \Omega}(u(y) - n|y - z|) \quad \text{for } z \in \Omega.$$

u_n is finite valued, since $M > u_n(z) \geq u(z_0) - n|z_0 - z|$.

We next prove that u_n is continuous: Given $z \in \Omega$ and $\epsilon > 0$ we choose $y \in \Omega$ such that

$$u_n(z) < u(y) - n|y - z| + \epsilon.$$

Then for any $z' \in \Omega$ we have

$$\begin{aligned} u_n(z) - u_n(z') &\leq (u(y) - n|y - z| + \epsilon) - (u(y) - n|y - z'|) \\ &= n(|z' - y| - |y - z|) + \epsilon \leq n|z - z'| + \epsilon \end{aligned}$$

and since $\epsilon > 0$ is arbitrary we set that

$$u_n(z) - u_n(z') \leq n|z - z'|.$$

Interchanging z by z' we get

$$|u_n(z) - u_n(z')| \leq n|z - z'|$$

proving the continuity.

Next note that

$$u(y) - n|y - z| \geq u(y) - (n + 1)|y - z|$$

which implies that $u_n \geq u_{n+1}$. Since

$$u_n(z) = \sup_{y\in\Omega}(u(y) - n|y - z|) \geq u(z)$$

it is left to prove that

$$\lim_{n\longrightarrow\infty} u_n(z) \leq u(z),$$

i.e. if $K > u(z)$ then $u_n(z) \leq K$ for sufficiently large n. For that we use that u is usc: There is a ball $B(z, \delta)$ such that $u(y) < K$ for all $y \in B(z, \delta)$. Now,

$$\begin{aligned} u_n(z) &= \sup_{y\in\Omega}(u(y) - n|y - z|) \\ &= \max\{\sup_{|y-z|<\delta}(u(y) - n|y - z|), \sup_{|y-z|\geq\delta}(u(y) - n|y - z|)\} \\ &\leq \max\{K, M - n\delta\} \end{aligned}$$

so if n is large then $u_n(z) \leq K$ as desired.

We shall finally deal with the general case in which u is no longer necessarily bounded from above. Choose an increasing homeomorphism

$$\phi : [-\infty, \infty] \longrightarrow [-\infty, 0],$$

(for example $\phi(t) = -\exp(-t)$). The function $\phi \circ u$ is usc and bounded from above, so the construction in the special case above yields a sequence $(v_n)_{n\geq 1}$ of continuous real valued functions with the property that $v_n(z) \downarrow \phi(u(z))$ as $n \longrightarrow \infty$ for all $z \in \Omega$. It suffices to verify that $-\infty < v_n(z) < 0$ for all $z \in \Omega$, because we may then choose $u_n := \phi^{-1} \circ v_n$.

The left inequality is trivial. To settle the other inequality we use the definition of v_n, i.e.

$$v_n(z) = \sup_{y\in\Omega}(\phi(u(y)) - n|y - z|).$$

By hypothesis $u(z) < \infty$, so $a := \phi(u(z)) < 0$, and by the upper semi-continuity of $\phi \circ u$ there exists $\delta > 0$ such that

$$\phi(u(y)) < \frac{a}{2} < 0 \quad \text{for all } y \in B(z, \delta).$$

Thus

$$\begin{aligned} v_n(z) &= \max\{\sup_{|y-x|<\delta}(\phi(u(y)) - n|y-z|), \sup_{|y-x|\geq\delta}(\phi(u(y)) - n|y-z|)\} \\ &\leq \max\{\tfrac{a}{2}, -n\delta\} < 0. \end{aligned}$$

□

Definition 1.3. *Let Ω be a non-empty open and connected plane set. A map $u : \Omega \longrightarrow [-\infty, \infty)$ is said to be* **subharmonic** *in Ω if*

a) u is upper semi-continuous and not identically $-\infty$.

b) For each $z \in \Omega$ there is a ball $B(z, R_z) \subseteq \Omega$ such that

$$u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta \quad \text{for all } 0 < r < R_z.$$

Proposition 1.3. *A function u which is subharmonic on Ω, is locally integrable in Ω. In particular, the set*

$$\{z \in \Omega \mid u(z) = \infty\}$$

is a null set and u cannot be identically $-\infty$ on any non-empty open subset of Ω.

Proof. We first observe that u is integrable over $B(a, R)$ if $\overline{B}(a, R) \subseteq \Omega$ and $u(a) > -\infty$:

Indeed, multiplying the inequality

$$u(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta$$

by πr and integrating from 0 to $R = R_a$ we get

$$-\infty < \frac{1}{2}\pi R^2 u(a) \leq \frac{1}{2}\int_0^R\int_0^{2\pi} u(a+re^{i\theta})d\theta r dr = \frac{1}{2}\int\int_{B(a,R)} u(x, y)dxdy.$$

As a consequence if u is not integrable at a then u must be identically $-\infty$ throughout some neighborhood of a. Thus the complement in Ω of the set

$$\{z \in \Omega \mid u \text{ is locally integrable at } z\}$$