بسم الله الرحمن الرحيم

In The Name of GOD

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پیوست

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This is to certify that Mrs. Fatemeh Helen Ghane defended successfully her Ph.D. thesis titled :

On chaotic behaviour and prevalence of Henon-like strange attractors on a family of endomorphisms of S .

Her thesis was evatuated as: Excellent

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Chaotic behavior in one dimension

by:

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Supervised by:

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INTRODUCTION

Homoclinic orbits, and the associated complexity, first were discovered by Poincare end descibed in his famouse work on the stability of the solar system around 1890 (see [P-1890]).

In 1935, Birkhoff showed that near a homoclinic orbit there is an extremely intricate complex of periodic solutions, mostly with a very high period (see [B-1935]). By the mid 1960, Smale was studied the complexity of homoclinic behavior and presented a very simple geometric example which was called horseshoe. It could be completely analysed and which showed all the complexity found before (see , [S-1965], [S-1967], [S-1970]). The Smale's result was improved by a number of mathematicians and used in hyperbolic dynamics. In particular they provided models for very complex (chaotic) dynamic behavior.

In fact, it was realized that the dynamics of a diffeomor phism shows a great complexity if and only if it has some hyperbolic periodic point with a homoclinic intersection of its stable and unstable manifolds. Moreover, chaotic dynamics always involves the presence of homoclinic orbits .

The notion of chaos in dynamical systems refers to a situation where (forward) orbits do not converge to a periodic or quasi-periodic orbit and the evolution of the orbits has some degree of unpredictability because their behaviour is sensitive with respect to initial conditions .One says that the dynamics is chaotic if this sensitivity holds for many orbits, in the sense of Lebesgue measure. Note that existence of homoclinic orbits alone is not sufficient to imply this. Homoclinic bifurcations, or non-transverse homoclinic orbits, became important when going beyond the hyperbolic theory.

In the late 1960's Newhouse combined homoclinic bifurcations with the complexity already available in the hyperbolic theory to obtain dynamical systems far more complicated than the hyperbolic ones . Ultimately this led to his famous result on the coexistence of infinitely many periodic attractors and was also influential on other work on hyperbolicty or lack of it near homoclinic bifurcations(see[N-1970], [N-1974], [N-1979]).

Homoclinic bifurcations are a main mechanism to unleash a string of complicated changes in the dynamics of a diffeomorphism or an endomorphism. Also a global description of nonhyperbolicity would follow from a good comprehension of homoclinic bifurcations and specially of the dynamic types occuring persistently in their unfolding.

The concept of persistence is essentialy measure theoric . One of these dynamical phenomena which occuring persistently on almost every family unfolding a homoclinic tangency is a Henon-like strange attractors. In 1976 [H - 1976], Henon performed a numerical study of the family of the plane.

$$h_{a,b}(x, y) = (1 - ax^2 + y, bx)$$

and observed for parameter values $a = 1.4, b = 0.3$ a nontrivial attractor with a highly intricate geometric structure. This family has been the sudject of (both numerical and theoretical) research and its dynamic is far from being completely understood . Benedicks and Carleson in their remarkable paper

[B.C-1991] showed that this family does exhibit a nonhyperbolic strange attractor for a positive Lebesgue measure set of parameter values close to a=2 and b=0 .It remains an open problem to prove the existence of a global strange attractor for the parameter initially considered by Henon, although implies that there are strange attractors with a more local nature for nearby parameters.

The Benedicks and Carleson's result has been extended by Mora and Viana [M.V-1993]: the Henon-like strange attractors appear whenever one generically unfolds a quadratic homoclinic tangency through one-parameter families of locally dissipative surface diffeomorphisms. Before this important work on the Henon-like strange attractors, a result for the one-dimensional map $f_a(x) = 1 - ax^2$ was obtaining by Jakobson [J-1981] and then by Benedicks and Carleson [B.C-1991]: the set of a-values for which f_a has positive Liapunov exponent on the critical orbit (chaotic behavior) has full density at bifurcation value $a = 2$.

There exist today many proofs of this theorem, as well as generalizations to families of smooth maps with finitely many critical points [T.T.Y-1992] and to families of maps in which a single discontinuity coincides with the critical point [R-1993]. Recently, Luzzatto and Viana, in an unpublished work, have introduced a class of one parameter families of real maps which extends classical geometric Lorenz models. Their result states that nonuniform expansion is the prevalence form of dynamics even after the formation of criticalities (see [L.V-to appear]). In this work we study a class of multimodal maps of the circle which appear naturally in the context of bifurcations of families of diffeomorphisms on manifolds. An important application of these maps is in the study of bifurcations via saddle-node critical cycles.

Indeed, it was observed by Newhouse, Palis and Takens [N.P.T-1983] that if a family of diffeomorphisms goes through a critical saddle-node cycle then conveniently defined return maps converge in a natural way to certain multimodal maps of the circle. This is precisely the class of maps we study here.

Critical saddle-node cycles are known to be a very powerful mechanism to create chaotic behviour ([see D.R.V-to appear]). In particular, the unfolding of such a cycle always involves homoclinic tangencies and all the complicated phenomenon associated to them, as proved by Newhouse, Palis and Takens [N.P.T-1983]. The converse is also true: whenever a homoclinic tangency is unfolded there is formation and destruction of critical saddle-node cycles, are observation due to Mora, see [P.T, 1993]. Therefore, the study of this class of one-dimensional multimodal maps specially their chaotic dynamics is an important and necessary step to understand these complex bifurcations of diffeomorphisms. The goal of this work is to carry out such a study.

In chapter one we will describe definitions and preliminary results to provide the global context of our own results to be presented in detail in the subsequent chapters.

In chapter two we consider degree-one maps of the circle and we study their rotation set. Our main result in this chapter says that if the map is topologically mixing then its rotation interval is nontrivial (that is, not reduced to a point) and nonpersistent (arbitary small pertubations may modify the rotation interval). Non-persistence is a very important property because, by a result of Newhouse, Palis and Takens [NP.T-1983], modification of the rotation interval always involve homoclinic bifurcations and so also formation of chaotic dynamics.

In chapter three we consider a class of one-parameter fmilies of degree-one maps of the circle and study the presence of strange attractors in these families. More precisely, we show that the derivative grows exponentially fast on the critical orbits for a set of parameter values with positive Lebesgue measure. Evenmore, this set of parameters has full Lebesgue density at the special bifurcation point. Then the closure of the critical orbits is a strange attractor for these maps.)

Chapter one

Definitions and preliminaries.

The purpose of this chapter is to collect a number of definitions and results which will be used in chapter 2 and 3. Before explaining the main ideas in more detail and state our results, let us give the precise definitions of the main notions involved. The study of the geometric structure of the orbits of dynamical systems (differential equations, flows, vector fields, diffeomorphisms or endomorphisms) defined on a manifold has been considered in many works since Poincare and Liapunov. Here, we consider endomorphisms on the circle in dimension one and diffeomorphisms in dimension two or higher dimension as dynamical systems and equip them with C^r-topology, in fact the space of differentiable systems of class $C^r, r \geq 1$, has a natural topology given by uniform convergence of the first r-derivatives.

Let M be a compact two-dimensional C^∞-manifold without boundary and I be an interval in real space.

A one-parameter family of diffeomorphisms (arc) is a C^r-map $\Phi : M \times I \to M \times I$, such that $\Phi(x, \mu) = (\varphi_\mu(x), \mu)$ where $x \to \varphi_\mu(x)$ is a C^r-diffeomorphism for each $\mu \in I$ and is denoted by (φ_μ).

Now, let $\varphi : M \to M$ be a C^2-diffeomorphism of a surface M and $p \in M$ be a fixed saddle, i.e $\phi(p) = p$ and $(d(\phi))_p$ has two real eigenvalues λ and σ with $0 < |\lambda| < 1 < |\sigma|$.

For simplicity we assume that $0 < \lambda < 1 < \sigma$. Observe that this can always be obtains replacing ϕ by ϕ^2 if necessary.

From the theory of hyperbolicity we know that:

-the stable and unstable separitrices of $p, W^s(p)$ and $W^u(p)$ are C^2;

-there are C^1-linearizing coordinates in a neighborhood U of p, i.e, C^1-linearizing coordinates x_1, x_2 such that $p = (0,0)$ and $\phi(x_1, x_2) = (\lambda x_1, \sigma x_2)$. This linearization follows at once from the existence of ϕ-invariant stable and unstable foliations near p which are of class C^1 (see [H-1964] and Appendix-1 of [P.T-1993]).