IN THE NAME OF GOD

SOME PROPERTIES OF COMPOSITION OPERATORS ON WEIGHTED HARDY SPACES

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Dedicated to

The memory of my parents

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ABSTRACT

SOME PROPERTIES OF COMPOSITION OPERATORS ON WEIGHTED HARDY SPACES

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Composition operators have been studied on a variety of spaces. The study of composition operators began with the work of Nordgren and Schwartz on classical Hardy space H^2.

Given a collection S of analytic functions on some domain and an analytic map φ from that domain into itself. We define the composition operator C_φ on S by $C_\varphi f = f \circ \varphi$ for $f \in S$. Our work is primarily devoted to the case where the collection S is the Weighted Hardy space $H^p(\beta)$, $1 < p < \infty$, of formal power series $f = \sum_{n=0}^{\infty} \hat{f}(n) z^n$, whose the coefficients satisfying in $\sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty$, and $\{\beta(n)\}_{n\geq 0}$ is a sequence of positive numbers with $\beta(0) = 1$. The weighted Hardy space $H^p(\beta)$ with $\|\cdot\|_\beta$,

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p$$

is a reflexive Banach space. The Hardy, Bergman and Dirichlet spaces can be viewed in this way when $p = 2$ and $\beta(n) = 1$, $\beta(n) = (n+1)^{\frac{-1}{2}}$ and $\beta(n) = (n+1)^{\frac{1}{2}}$, respectively. We study the relationship betwーen properties of C_φ and properties of the symbol φ. The goal is to see boundedness, compactness and Fredholm of C_φ as a consequence of particular geometric and analytic features of the function φ.

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1. BACKGROUND

Introduction

Given a collection S of analytic functions on some domain and an analytic map φ from that domain into itself. We define the composition operator C_φ on S by $C_\varphi f = f \circ \varphi$ for $f \in S$. Without saying anything more, there is no reason that $C_\varphi f$ should even belong to S. However, in many cases where the collection S is a Banach space, the operator C_φ maps S into itself.

Composition operators have been studied on a variety of spaces. The study of composition operators began with the work of Nordgren [19] and Schwartz [20] on classical Hardy space H^2. The first explicit reference to composition operators in the Mathematical Subject Classification Index appeared in 1990.

Our work is primarily devoted to the case where the collection S is the weighted Hardy spaces $H^p(\beta)$, $1 < p < \infty$, of formal power series $f = \sum_{n=0}^{\infty} \hat{f}(n) z^n$, whose coefficients satisfying in $\sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty$, for some weight $\{\beta(n)\}_{n \geq 0}$.

Section 1

Classical Banach Spaces of Analytic Functions

In this section we begin by defining the Hardy spaces of the unit disc $D = \{z : |z| < 1\}$ in the complex plane. For more details one can refer to the books [9],[10],[11],[15].

Definition 1.1.1. For $0 < p < \infty$ the *Hardy space* H^p is the space of all functions f analytic in D such that $\|f\|_p^p = \sup\limits_{0 \le r \le 1} (\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta) < \infty$.

For $1 \le p < \infty$, the functional $\| \cdot \|_p$ is a norm which makes H^p into a Banach space, while for $0 < p < 1$, the metric

$$d(f,g) = \|f - g\|_p^p$$

makes H^p into a complete linear metric space.

For $p = \infty$, H^∞ is the Banach space of bounded analytic functions on D, taken in the supremum norm:

$$\|f\|_\infty = \sup_{z \in D} |f(z)|.$$

For $p = 2$, H^2 is a Hilbert space with the inner product

$$< f, g >= \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})} \frac{d\theta}{2\pi}$$

and an easy calculation shows that the monomials $1, z, z^2, \ldots$ form an orthonormal set. Since H^2 is spanned by the monomials, $\{1, z, z^2, \ldots\}$ is an orthonormal basis for H^2, and we regard this as the standard basis. Since every function analytic on the open disc has a MacLaurin expansion that converges absolutely and uniformly on compact subsets of D, for $f(z) = \sum_{n\geq 0} \hat{f}(n)z^n$ we have

$$\begin{aligned}
\|f\|_2^2 &= \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} \\
&= \sup_{0<r<1} \int_0^{2\pi} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \hat{f}(n)\overline{\hat{f}(k)} r^{n+k} e^{i(n-k)\theta} \frac{d\theta}{2\pi} \\
&= \sup_{0<r<1} \sum_{n=0}^{\infty} |\hat{f}(n)|^2 r^{2n} = \sum_{n=0}^{\infty} |\hat{f}(n)|^2.
\end{aligned}$$

By this calculation we have:

$$H^2 = \{f = \sum_{n=0}^{\infty} \hat{f}(n)z^n : \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty\}.$$

Now consider linear transformation M_z of multiplication by z on H^2 defined by

$$(M_z f)(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^{n+1}.$$

So $(M_z f\hat{)}(n) = \begin{cases} \hat{f}(n-1) & n \geq 1 \\ 0 & n = 0. \end{cases}$

Clearly M_z shifts the standard basis. Easy calculation shows that the forward unilateral shift, defined by

$$S(\hat{f}(0), \hat{f}(1), \hat{f}(2), \ldots) = (\hat{f}(1), \hat{f}(2), \ldots)$$

on $l^2(\mathbb{N})$ is unitarily equivalent to the operator of multiplication by z on H^2.

Definition 1.1.2. For $0 < p < \infty$, the *Bergman space* A^p is the set of functions analytic on the unit disc for which

$$\int_D |f(z)|^p \frac{dA(z)}{\pi} < \infty,$$

where $dA(z)$ is the Lebesgue area measure on the unit disc and

$$\|f\|_p = \{\frac{1}{\pi}\int_D |f(z)|^p dA(z)\}^{\frac{1}{p}}.$$

For $p \geq 1$, A^p is a Banach space with norm $\|f\|_p$. For $0 < p < 1$, A^p is a non-locally convex topological vector space and $d(f,g) = \|f-g\|_p^p$ is a complete metric on A^p.

Moreover, A^2 is a Hilbert space with inner product

$$< f, g >= \frac{1}{\pi} \int f(z)\overline{g(z)} dA(z).$$

A calculation using polar coordinates shows that the monomials $1, z, z^2, \dots$ are orthogonal in A^2 and $\|z^n\|^2 = \frac{1}{n+1}$. For MacLaurin expansion $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ and $\rho < 1$ we have:

$$\begin{aligned}
\int_{\rho D} |f(z)|^2 \frac{dA(z)}{\pi} &= \int_0^{\rho} \int_0^{2\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \hat{f}(n) \overline{\hat{f}(k)} r^{n+k+1} e^{i(n-k)\theta} \frac{dr d\theta}{\pi} \\
&= \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \int_0^{\rho} 2r^{2n+1} dr \\
&= \sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{n+1} \rho^{2n+2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|f\|^2 &= \lim_{\rho \to 1} \int_{\rho D} |f(z)|^2 \frac{dA(z)}{\pi} \\
&= \lim_{\rho \to 1} \sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{n+1} \rho^{2n+1} \\
&= \sum_{n \geq 0} \frac{|\hat{f}(n)|^2}{n+1}.
\end{aligned}$$

So we can characterize the Bergman space by

$$A^2 = \{ f = \sum_{n=0}^{\infty} \hat{f}(n) z^n : \sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{n+1} < \infty \}.$$

For complete discussions of the Bergman spaces see [31],[1].

Definition 1.1.3. The set of functions analytic on the unit disc for which

$$\int_D |f'(z)|^2 \frac{dA(z)}{\pi} < \infty$$

with norm given by

$$\|f\|_D^2 = |f(0)|^2 + \int_D |f'(z)|^2 \frac{dA(z)}{\pi}$$

and inner product

$$< f, g >_D = f(0)\overline{g(0)} + \int_D f'(z)\overline{g'(z)} \frac{dA(z)}{\pi}$$

is called the *Dirichlet space* $\mathbb{D}$.

We can see that the monomials $1, z, z^2, \ldots$ form an orthogonal basis for $\mathbb{D}$. For MacLaurin expansion, $f(z) = \sum\limits_{n=0}^{\infty} \hat{f}(n) z^n$ and $\rho < 1$ we have:

$$\begin{aligned}
\int_{\rho D} |f'(z)|^2 \frac{dA(z)}{\pi} &= \int_0^{\rho} \int_0^{2\pi} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \hat{f}(n) \overline{\hat{f}(k)} r^{n+k-1} e^{i(n-k)\theta} \frac{dr\, d\theta}{\pi} \\
&= \sum_{n=1}^{\infty} n^2 |\hat{f}(n)|^2 \frac{\rho^{2n}}{n} \\
&= \sum_{n=1}^{\infty} n |\hat{f}(n)|^2 \rho^{2n}.
\end{aligned}$$

So

$$\begin{aligned}
\|f\|_D &= |f(0)|^2 + \lim_{\rho \to 1} \sum_{n=1}^{\infty} n |\hat{f}(n)|^2 \rho^{2n} \\
&= |f(0)|^2 + \sum_{n=1}^{\infty} n |\hat{f}(n)|^2 \\
&= \sum_{n=0}^{\infty} (n+1) |\hat{f}(n)|^2.
\end{aligned}$$

Definition 1.1.4. A Banach space of analytic complex valued functions on a set X that the vector operations are the pointwise operations, $f(x) = g(x)$ for each $x \in X$ then $f = g$, $f(x) = f(y)$ for each function in the space implies $x = y$ and for each x in X, the linear functional $f \mapsto f(x)$ is continuous, is called a functional Banach space of analytic functions.

Let K_x be the linear fucntional for evaluation at x, that is, $K_x(f) = f(x)$. For functional Hilbert space, by the Rieze representation theorem, corresponding to each $x \in X$ there is a function, call K_x, in the Hilbert space that induces this linear functional, $f(x) =< f, K_x >$. In this case the functions K_x are called the ***reproducing kernels.***

Multiplication operators characterized by their adjoints having the K_x's an eigen vectors [26]. Composition operators, also can be characterized by considering the set of point evaluation linear functionals. Caughran and Schwartz [2] showed that:

Theorem 1.1.5. An operator A on Y, functional Banach space of X into itself, is a composition operator if and only if the set $\{K_x : x \in X\}$ is invariant under A^*. In this case φ is determined by $A^*K_x = K_{\varphi(x)}$ and $A = C_\varphi$.

Proof. If $A = C_\varphi$ then for every f in Y,

$$(f, A^*K_x) = (Af, K_x) = f(\varphi(x)) = (f, K_{\varphi(x)})$$

so $A^*K_x = K_{\varphi(x)}$. Conversely, if $A^*K_x = K_{\varphi(x)}$ then

$$Af(x) = (Af, K_x) = (f, K_{\varphi(x)}) = f(\varphi(x)).$$

Hence $A = C_\varphi$. $\square$

By definition of classical Hardy, Bergman and Dirichlet space evaluation at λ in the disc on H^2, A^2 and $\mathbb{D}$ given by $f(\lambda) = (f, K_\lambda)$, where on $H^2(D)$ we have:

$$K_\lambda(z) = \frac{1}{1-\bar{\lambda}z} \quad \text{and} \quad \|K_\lambda\| = \frac{1}{\sqrt{1-|\lambda|^2}}$$

and on $A^2(D)$ we have:

$$K_\lambda(z) = \frac{1}{(1-\bar{\lambda}z)^2} \quad \text{and} \quad \|K_\lambda\| = \frac{1}{1-|\lambda|^2}$$

and on $\mathbb{D}$ we have:

$$K_\lambda(z) = \frac{1}{\bar{\lambda} z}\log(\frac{1}{1-\bar{\lambda} z}) \quad \text{and} \quad \|K_\lambda\|^2 = \frac{1}{|\lambda|^2}\log(\frac{1}{1-|\lambda|^2}).$$

Since polynomials are dense in the Hardy and Bergman Banach spaces for $1 \leq p < \infty$, we can get another formula for the evaluation functional as follows. If $f \in H^p$ and $|\lambda| < 1$ then

$$f(\lambda) = \int_0^{2\pi} \frac{f(e^{i\theta})}{1-\lambda e^{-i\theta}}\frac{d\theta}{2\pi}$$

and

$$\|K_\lambda\| = (\frac{1}{1-|\lambda|^2})^{\frac{1}{p}}.$$

If $f \in A^p$ and $|\lambda| < 1$ then

$$f(\lambda) = \int_D \frac{f(z)}{(1-\lambda\bar{z})^2}\frac{dA(z)}{\pi}.$$

For a proof and more details can refer to [8].