

IN THE NAME OF GOD

**SOME PROPERTIES OF COMPOSITION
OPERATORS ON WEIGHTED HARDY
SPACES**

BY:

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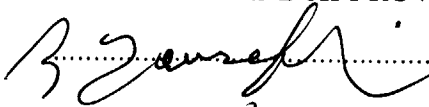
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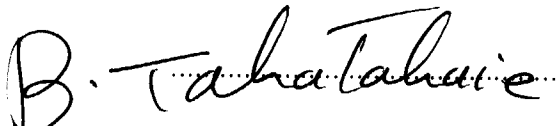
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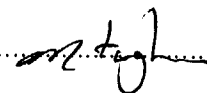
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Dedicated to

The memory of my parents

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ABSTRACT

SOME PROPERTIES OF COMPOSITION OPERATORS ON WEIGHTED HARDY SPACES

BY:

SEDIGHEH JAHEDI

Composition operators have been studied on a variety of spaces. The study of composition operators began with the work of Nordgren and Schwartz on classical Hardy space H^2 .

Given a collection S of analytic functions on some domain and an analytic map φ from that domain into itself. We define the composition operator C_φ on S by $C_\varphi f = f \circ \varphi$ for $f \in S$. Our work is primarily devoted to the case where the collection S is the Weighted Hardy space $H^p(\beta)$, $1 < p < \infty$, of formal power series $f = \sum_{n=0}^{\infty} \hat{f}(n)z^n$, whose the coefficients satisfying in $\sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty$, and $\{\beta(n)\}_{n \geq 0}$ is a sequence of positive numbers with $\beta(0) = 1$. The weighted Hardy space $H^p(\beta)$ with $\|\cdot\|_\beta$,

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p$$

is a reflexive Banach space. The Hardy, Bergman and Dirichlet spaces can be viewed in this way when $p = 2$ and $\beta(n) = 1$, $\beta(n) = (n + 1)^{-\frac{1}{2}}$ and $\beta(n) = (n + 1)^{\frac{1}{2}}$, respectively. We study the relationship between properties of C_φ and properties of the symbol φ . The goal is to see boundedness, compactness and Fredholm of C_φ as a consequence of particular geometric and analytic features of the function φ .

TABLE OF CONTENTS

CONTENTS	PAGE
CHAPTER I: BACKGROUND	1
Section 1. Classical Banach Spaces of analytic functions	3
Section 2. Geometric Function Theory	9
CHAPTER II: WEIGHTED SEQUENCE SPACES	17
Section 1. Elementary Properties	18
Section 2. Bounded Point Evaluations	22
Section 3. Analytic Structure of Weighted Hardy Spaces	25
Section 4. Multipliers and Some Theorems	30
CHAPTER III: BOUNDEDNESS OF COMPOSITION OPERATORS	38
Section 1. Boundedness in Classical Hardy Spaces	39
Section 2. Boundedness in Weighted Hardy Spaces	42
CHAPTER IV: COMPACTNESS OF COMPOSITION OPERATORS	52
Section 1. Compactness on Classical Hardy Spaces	53
Section 2. Compactness and Essential Norm on Weighted	60

	Hardy Spaces	
Section 3.	Compactness and $\ \varphi\ _\infty < 1$	74
Section 4.	Compactness and Fixed Points	79
REFERENCES		84
ABSTRACT AND TITLE PAGE IN PERSIAN		

CHAPTER I
BACKGROUND

1. BACKGROUND

Introduction

Given a collection S of analytic functions on some domain and an analytic map φ from that domain into itself. We define the composition operator C_φ on S by $C_\varphi f = f \circ \varphi$ for $f \in S$. Without saying anything more, there is no reason that $C_\varphi f$ should even belong to S . However, in many cases where the collection S is a Banach space, the operator C_φ maps S into itself.

Composition operators have been studied on a variety of spaces. The study of composition operators began with the work of Nordgren [19] and Schwartz [20] on classical Hardy space H^2 . The first explicit reference to composition operators in the Mathematical Subject Classification Index appeared in 1990.

Our work is primarily devoted to the case where the collection S is the weighted Hardy spaces $H^p(\beta)$, $1 < p < \infty$, of formal power series $f = \sum_{n=0}^{\infty} \hat{f}(n)z^n$, whose coefficients satisfying in $\sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty$, for some weight $\{\beta(n)\}_{n \geq 0}$.

Section 1

Classical Banach Spaces of Analytic Functions

In this section we begin by defining the Hardy spaces of the unit disc $D = \{z : |z| < 1\}$ in the complex plane. For more details one can refer to the books [9],[10],[11],[15].

Definition 1.1.1. For $0 < p < \infty$ the *Hardy space* H^p is the space of all functions f analytic in D such that $\|f\|_p^p = \sup_{0 \leq r \leq 1} (\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta) < \infty$.

For $1 \leq p < \infty$, the functional $\|\cdot\|_p$ is a norm which makes H^p into a Banach space, while for $0 < p < 1$, the metric

$$d(f, g) = \|f - g\|_p^p$$

makes H^p into a complete linear metric space.

For $p = \infty$, H^∞ is the Banach space of bounded analytic functions on D , taken in the supremum norm:

$$\|f\|_\infty = \sup_{z \in D} |f(z)|.$$

For $p = 2$, H^2 is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \frac{d\theta}{2\pi}$$

and an easy calculation shows that the monomials $1, z, z^2, \dots$ form an orthonormal set. Since H^2 is spanned by the monomials, $\{1, z, z^2, \dots\}$ is an orthonormal basis for H^2 , and we regard this as the standard basis. Since every function analytic on the open disc has a MacLaurin expansion that converges

absolutely and uniformly on compact subsets of D , for $f(z) = \sum_{n \geq 0} \hat{f}(n)z^n$ we have

$$\begin{aligned} \|f\|_2^2 &= \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} \\ &= \sup_{0 < r < 1} \int_0^{2\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \hat{f}(n) \overline{\hat{f}(k)} r^{n+k} e^{i(n-k)\theta} \frac{d\theta}{2\pi} \\ &= \sup_{0 < r < 1} \sum_{n=0}^{\infty} |\hat{f}(n)|^2 r^{2n} = \sum_{n=0}^{\infty} |\hat{f}(n)|^2. \end{aligned}$$

By this calculation we have:

$$H^2 = \left\{ f = \sum_{n=0}^{\infty} \hat{f}(n)z^n : \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty \right\}.$$

Now consider linear transformation M_z of multiplication by z on H^2 defined by

$$(M_z f)(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^{n+1}.$$

$$\text{So } (M_z f)\hat{(n)} = \begin{cases} \hat{f}(n-1) & n \geq 1 \\ 0 & n = 0. \end{cases}$$

Clearly M_z shifts the standard basis. Easy calculation shows that the forward unilateral shift, defined by

$$S(\hat{f}(0), \hat{f}(1), \hat{f}(2), \dots) = (\hat{f}(1), \hat{f}(2), \dots)$$

on $l^2(\mathbb{N})$ is unitarily equivalent to the operator of multiplication by z on H^2 .

Definition 1.1.2. For $0 < p < \infty$, the *Bergman space* A^p is the set of functions analytic on the unit disc for which

$$\int_D |f(z)|^p \frac{dA(z)}{\pi} < \infty,$$

where $dA(z)$ is the Lebesgue area measure on the unit disc and

$$\|f\|_p = \left\{ \frac{1}{\pi} \int_D |f(z)|^p dA(z) \right\}^{\frac{1}{p}}.$$

For $p \geq 1$, A^p is a Banach space with norm $\|f\|_p$. For $0 < p < 1$, A^p is a non-locally convex topological vector space and $d(f, g) = \|f - g\|_p^p$ is a complete metric on A^p .

Moreover, A^2 is a Hilbert space with inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int f(z) \overline{g(z)} dA(z).$$

A calculation using polar coordinates shows that the monomials $1, z, z^2, \dots$ are orthogonal in A^2 and $\|z^n\|^2 = \frac{1}{n+1}$. For MacLaurin expansion $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ and $\rho < 1$ we have:

$$\begin{aligned} \int_{\rho D} |f(z)|^2 \frac{dA(z)}{\pi} &= \int_0^\rho \int_0^{2\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \hat{f}(n) \overline{\hat{f}(k)} r^{n+k+1} e^{i(n-k)\theta} \frac{dr d\theta}{\pi} \\ &= \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \int_0^\rho 2r^{2n+1} dr \\ &= \sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{n+1} \rho^{2n+2}. \end{aligned}$$

It follows that

$$\begin{aligned} \|f\|^2 &= \lim_{\rho \rightarrow 1} \int_{\rho D} |f(z)|^2 \frac{dA(z)}{\pi} \\ &= \lim_{\rho \rightarrow 1} \sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{n+1} \rho^{2n+2} \\ &= \sum_{n \geq 0} \frac{|\hat{f}(n)|^2}{n+1}. \end{aligned}$$

So we can characterize the Bergman space by

$$A^2 = \left\{ f = \sum_{n=0}^{\infty} \hat{f}(n) z^n : \sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{n+1} < \infty \right\}.$$

For complete discussions of the Bergman spaces see [31],[1].

Definition 1.1.3. The set of functions analytic on the unit disc for which

$$\int_D |f'(z)|^2 \frac{dA(z)}{\pi} < \infty$$

with norm given by

$$\|f\|_D^2 = |f(0)|^2 + \int_D |f'(z)|^2 \frac{dA(z)}{\pi}$$

and inner product

$$\langle f, g \rangle_D = f(0)\overline{g(0)} + \int_D f'(z)\overline{g'(z)} \frac{dA(z)}{\pi}$$

is called the *Dirichlet space* \mathbb{D} .

We can see that the monomials $1, z, z^2, \dots$ form an orthogonal basis for \mathbb{D} . For MacLaurin expansion, $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ and $\rho < 1$ we have:

$$\begin{aligned} \int_{\rho D} |f'(z)|^2 \frac{dA(z)}{\pi} &= \int_0^\rho \int_0^{2\pi} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \hat{f}(n)\overline{\hat{f}(k)} r^{n+k-1} e^{i(n-k)\theta} \frac{dr d\theta}{\pi} \\ &= \sum_{n=1}^{\infty} n^2 |\hat{f}(n)|^2 \frac{\rho^{2n}}{n} \\ &= \sum_{n=1}^{\infty} n |\hat{f}(n)|^2 \rho^{2n}. \end{aligned}$$

So

$$\begin{aligned} \|f\|_D &= |f(0)|^2 + \lim_{\rho \rightarrow 1} \sum_{n=1}^{\infty} n |\hat{f}(n)|^2 \rho^{2n} \\ &= |f(0)|^2 + \sum_{n=1}^{\infty} n |\hat{f}(n)|^2 \\ &= \sum_{n=0}^{\infty} (n+1) |\hat{f}(n)|^2. \end{aligned}$$

Definition 1.1.4. A Banach space of analytic complex valued functions on a set X that the vector operations are the pointwise operations, $f(x) = g(x)$ for each $x \in X$ then $f = g$, $f(x) = f(y)$ for each function in the space implies $x = y$ and for each x in X , the linear functional $f \mapsto f(x)$ is continuous, is called a functional Banach space of analytic functions.

Let K_x be the linear functional for evaluation at x , that is, $K_x(f) = f(x)$. For functional Hilbert space, by the Rieze representation theorem,

corresponding to each $x \in X$ there is a function, call K_x , in the Hilbert space that induces this linear functional, $f(x) = \langle f, K_x \rangle$. In this case the functions K_x are called the *reproducing kernels*.

Multiplication operators characterized by their adjoints having the K_x 's as eigen vectors [26]. Composition operators, also can be characterized by considering the set of point evaluation linear functionals. Caughran and Schwartz [2] showed that:

Theorem 1.1.5. An operator A on Y , functional Banach space of X into itself, is a composition operator if and only if the set $\{K_x : x \in X\}$ is invariant under A^* . In this case φ is determined by $A^*K_x = K_{\varphi(x)}$ and $A = C_\varphi$.

Proof. If $A = C_\varphi$ then for every f in Y ,

$$(f, A^*K_x) = (Af, K_x) = f(\varphi(x)) = (f, K_{\varphi(x)})$$

so $A^*K_x = K_{\varphi(x)}$. Conversely, if $A^*K_x = K_{\varphi(x)}$ then

$$Af(x) = (Af, K_x) = (f, K_{\varphi(x)}) = f(\varphi(x)).$$

Hence $A = C_\varphi$. \square

By definition of classical Hardy, Bergman and Dirichlet space evaluation at λ in the disc on H^2 , A^2 and \mathbf{D} given by $f(\lambda) = (f, K_\lambda)$, where on $H^2(D)$ we have:

$$K_\lambda(z) = \frac{1}{1 - \bar{\lambda}z} \quad \text{and} \quad \|K_\lambda\| = \frac{1}{\sqrt{1 - |\lambda|^2}}$$

and on $A^2(D)$ we have:

$$K_\lambda(z) = \frac{1}{(1 - \bar{\lambda}z)^2} \quad \text{and} \quad \|K_\lambda\| = \frac{1}{1 - |\lambda|^2}$$

and on \mathbb{D} we have:

$$K_\lambda(z) = \frac{1}{\bar{\lambda}z} \log\left(\frac{1}{1 - \bar{\lambda}z}\right) \quad \text{and} \quad \|K_\lambda\|^2 = \frac{1}{|\lambda|^2} \log\left(\frac{1}{1 - |\lambda|^2}\right).$$

Since polynomials are dense in the Hardy and Bergman Banach spaces for $1 \leq p < \infty$, we can get another formula for the evaluation functional as follows. If $f \in H^p$ and $|\lambda| < 1$ then

$$f(\lambda) = \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - \lambda e^{-i\theta}} \frac{d\theta}{2\pi}$$

and

$$\|K_\lambda\| = \left(\frac{1}{1 - |\lambda|^2}\right)^{\frac{1}{p}}.$$

If $f \in A^p$ and $|\lambda| < 1$ then

$$f(\lambda) = \int_D \frac{f(z)}{(1 - \lambda\bar{z})^2} \frac{dA(z)}{\pi}.$$

For a proof and more details can refer to [8].